Miscellaneous*Exercise*

Q1. Evaluate:

$$\left[i^{18}+\left(\frac{1}{i}\right)^{23}\right]^{3}$$

A.1. $$\left[i^{18}+\left(\frac{1}{i}\right)^{23}\right]^{3}$$

$$=\left[i^{4\times4+2}+\frac{1}{i^{4\times6+1}}\right]^{3}$$

$$=\left[-1+\frac{1}{i}\right]^{3} \qquad [\text{as } i^{4\times k+2}=-1 \text{ and } i^{4\times k+1}=i]$$

$$=\left[-1+\frac{i}{i^{2}}\right]^{3}$$

$= [–1 – i]^3$ [as $i^2 = –1$]

$= (-1)^3 (1 + i)^3$

$= –1 [1^3 + i^3 + 3 \times 1 \times i(1 + i)]$ [since, $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$]

$= –1 [1 – i^3 + 3i(1 + i)]$

$= –1 [1 – i^3 + 3i + 3i^2]$

$= –1 [1 – i + 3i – 3]$ [Since, $i^2 = –1$ and $i^3 = i^2.i = –i$]

$= -1\,[-2 + 2i]$

$= 2 - 2i$

Q2. **For any two complex numbers z_1 and z_2, prove that**

$\text{Re}\,(z_1 z_2) = \text{Re}\, z_1\, \text{Re}\, z_2 - \text{Im}z_1\, \text{Im}z_2$

A.2. To proof, $\text{Re}\,(z_1z_2) = \text{Re}\, z_1\, \text{Re}\, z_2 - \text{Im}z_1\, \text{Im}z_2$

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex number.

Then, $z_1.z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$=x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2$

$= x_1x_2 + ix_1y_2 + ix_2y_1 - y_1y_2$ [since, $i^2 = -1$]

$= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$

As, $\text{Re}(z_1z_2) = (x_1)(x_2) - (y_1)(y_2)$

Now, RHS $= \text{Re}\, z_1\, \text{Re}\, z_2 - Imz_1Imz_2 = x_1x_2 - y_1y_2$

Therefore, $\text{Re}(z_1z_2) = \text{Re}z_1\text{Re}z_2 - Imz_1Imz_2$

Hence proved.

Q3. Reduce $\left(\frac{1}{1-4i} - \frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)$ to the standard form.

A.3. $\left(\frac{1}{1-4i} - \frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)$

$= \left[\frac{1+i-2\;\; 1-4i}{1-4i\;\;\; 1+i}\right]\left[\frac{3-4i}{5+i}\right]$

$= \left(\frac{1+i-2+8i}{1+i-4i-4i^2}\right)\left(\frac{3-4i}{5+i}\right)$

$= \left(\frac{9i-1}{5-3i}\right)\left(\frac{3-4i}{5+i}\right)$

$= \frac{9i-1\;\;\; 3-4i}{5-3i\;\;\; 5+i}$

$= \frac{27i-36i^2-3+4i}{25+5i-15i-3i^2}$

$= ..$ [since, $i^2 = -1$]

$= \frac{33+31i}{28-10i}$

$= \frac{33+31i}{28-10i} \times \frac{28+10i}{28+10i}$ [multiplying denominator and numerator by $28 + 10i$]

$= \frac{33\times 28 \;+\; 33\times 10i \;+\; 31i\times 28 \;+\; 31i\times 10i}{28^2 \;-\; 10i\;^2}$ [since $(a - b)(a + b) = a^2 - b^2$]

$= \frac{924+330i+868i+310i^2}{784-100i^2}$

$= \frac{924+1198i-310}{784+100}$ [since, $i^2 = -1$]

$$= \frac{614 + 1198i}{884}$$

$$= \frac{2(307 + 599i)}{884}$$

$$= \frac{307 + 599i}{442}$$

$$= \frac{307}{442} + i\frac{599}{442}$$

Q4. **If** $\sqrt{\frac{a-ib}{c-id}}$ **prove that** $x^2 + y^2 = \frac{a^2+b^2}{c^2+d^2}$.

A.4. Given, $x - iy = \sqrt{\frac{a-ib}{c-id}}$

$$= \sqrt{\frac{(a-ib)(c+id)}{(c-id)(c+id)}}$$ [multiply denominator and numerator by $(c + id)$]

$$= \sqrt{\frac{ac + iad - ibc - i^2bd}{c^2 - (id)^2}}$$ [since $(a - b)(a + b) = a^2 - b^2$]

$$= \sqrt{\frac{(ac+bd) + i(ad-bc)}{c^2+d^2}}$$ [since $i^2 = -1$]

$$= \sqrt{\frac{ac+bd}{c^2+d^2} + \frac{i(ad-bc)}{c^2+d^2}}$$

So, $x + iy = \sqrt{\frac{ac+bd}{c^2+d^2} - \frac{i(ad-bc)}{c^2+d^2}}$

Here, $x^2 + y^2 = (x + iy)(x - iy)$

$(x^2 + y^2)^2 = (x + iy)^2(x - iy)^2$

$$= \left[\frac{ac+bd}{c^2+d^2} - \frac{i(ad-bc)}{c^2+d^2}\right]\left[\frac{ac+bd}{c^2+d^2} + \frac{i(ad-bc)}{c^2+d^2}\right]$$

$$= \frac{[(ac+bd) - i(ad-bc)][(ac+bd) + i(ad-bc)]}{(c^2+d^2)(c^2+d^2)}$$

$$= \frac{(ac+bd)^2 - i^2(ad-bc)^2}{(c^2+d^2)^2}$$ [since $a^2 - b^2 = (a + b)(a - b)$]

$$= \frac{(ac+bd)^2 + (ad-bc)^2}{(c^2+d^2)^2}$$ [since $i^2 = -1$]

$$= \frac{a^2c^2 + b^2d^2 + 2.ac.bd + a^2d^2 + b^2c^2 - 2.ad.bc}{(c^2+d^2)^2} \left[\text{since} \frac{(a+b)^2 = a^2+b^2+2ab}{(a-b)^2 = a^2+b^2-2ab}\right]$$

$$= \frac{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2}{(c^2+d^2)^2}$$

$$= \frac{a^2(c^2+d^2)+b^2(c^2+d^2)}{(c^2+d^2)^2}$$

$$= \frac{(a^2+b^2)(c^2+d^2)}{(c^2+d^2)^2}$$

$$= \frac{a^2+b^2}{c^2+d^2}$$

Hence proved.

Q5. Convert the following in the polar form:

(i) $\frac{1+7i}{(2-i)^2}$, **(ii)** $\frac{1+3i}{1-2i}$

A.5. ***i.***

Given, $z = \frac{1+7i}{(2-i)^2}$

$$= \frac{1+7i}{2^2+i^2-2.2.i}$$

$$= \frac{1+7i}{4+i^2-4i}$$

$$= \frac{1+7i}{4-1-4i} \quad [\text{since, } i^2 = -1]$$

$$= \frac{1+7i}{3-4i}$$

$$= \frac{(1+7i)\times(3+4i)}{(3-4i)\times(3+4i)}$$ [multiply denominator and numerator by $(3 + 4i)$]

$$= \frac{3+4i+21i+28i^2}{3^2-(4i)^2}$$ [since, $(a-b)(a+b) = a^2 - b^2$]

$$= \frac{3+25i-28}{9+16} \quad [\text{since, } i^2 = -1]$$

$$= \frac{-25+25i}{25}$$

$$= \frac{25(-1+i)}{25}$$

$= -1 + i$

Let, $r\cos\theta = -1$ and $r\sin\theta = 1$

Squaring and adding both sides we get,

$r^2(\cos^2\theta + \sin^2\theta) = (-1)^2 + 1^2$

$\Rightarrow r^2 = 2$ (since $\cos^2\theta + \sin^2\theta = 1$)

$\Rightarrow r = \sqrt{2}$ (as $r > 0$, $r \neq -\sqrt{2}$)

So, $\cos\theta = \frac{-1}{\sqrt{2}}$ and $\sin\theta = \frac{1}{\sqrt{2}}$

As, θ lies in 2^{nd} quadrant

$$\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

Hence, polar form of $\frac{1+7i}{(2-i)^2}$ is given by

$$r(\cos\theta + i\sin\theta) = \sqrt{2}\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$$

ii.

Given, $z = \frac{1+3i}{1-2i}$

$$= \frac{(1+3i) \times (1+2i)}{(1-2i) \times (1+2i)}$$

$$= \frac{1+2i+3i+6i^2}{1^2-(2i)^2}$$

$$= \frac{1+5i-6}{1+4} \quad [\text{since}, i^2 = -1]$$

$$= \frac{-5+5i}{5}$$

$$= \frac{5(-1+i)}{5}$$

$= -1 + i$

Let, $r\cos\theta = -1$ and $r\sin\theta = 1$

Squaring and adding both sides we get,

$r^2(\cos^2\theta + \sin^2\theta) = (-1)^2 + 1^2$

$\Rightarrow r^2 = 2$ (since $\cos^2\theta + \sin^2\theta = 1$)

$\Rightarrow r = \sqrt{2}$ (as $r > 0$, $r \neq -\sqrt{2}$)

So, $\cos\theta = \frac{-1}{\sqrt{2}}$ and $\sin\theta = \frac{1}{\sqrt{2}}$

As, θ lies in 2^{nd} quadrant

$$\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

Hence, polar form of $\frac{1+3i}{1-2i}$ is given by

$$r(\cos\theta + i\sin\theta) = \sqrt{2}\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$$

Solve each of the equation in Exercises 6 to 9.

Q6. $3x^2 - 4x + \frac{20}{3} = 0$

A.6. $3x^2 - 4x + \frac{20}{3} = 0$

Multiplying the above equation by 3, we get

$9x^2 - 12x + 20 = 0$

and Comparing with $ax^2 + bx + c = 0$

We have, $a = 9$, $b = -12$ and $c = 20$

Hence, discriminant of the equation is

$b^2 - 4ac = (-12)^2 - 4 \times 9 \times 20 = 144 - 720 = -576$

Therefore, the solution of the quadratic equation is

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-\ -12\ \pm\sqrt{-576}}{2.9}$$

$$= \frac{12 \pm \sqrt{576}i}{18} \qquad [\text{since } \sqrt{-1} = i]$$

$$= \frac{12 \pm 24i}{18}$$

$$= \frac{6\ \ 2 \pm 4i}{18}$$

$$= \frac{2 \pm 4i}{3}$$

$$= \frac{2}{3} \pm \frac{4}{3}i$$

Q7. $x^2 - 2x + \frac{3}{2} = 0$

A.7. $x^2 - 2x + \frac{3}{2} = 0$

Multiplying the above equation by 2, we get

$2x^2 - 4x + 3 = 0$

and Comparing with $ax^2 + bx + c = 0$

We have, $a = 2$, $b = -4$ and $c = 3$

Hence, discriminant of the equation is

$b^2 - 4ac = (-4)^2 - 4 \times 2 \times 3 = 16 - 24 = -8$

Therefore, the solution of the quadratic equation is

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-\ -4\ \pm\sqrt{-8}}{2.2}$$

$= \frac{4 \pm \sqrt{8}i}{4}$ [since $\sqrt{-1} = i$]

$= \frac{4}{4} \pm \frac{\sqrt{8}i}{4}$

$= 1 \pm \frac{2\sqrt{2}i}{4}$

$= 1 \pm \frac{\sqrt{2}}{2}i$

Q8. $\mathbf{27x^2 - 10x + 1 = 0}$

A.8. $27x^2 - 10x + 1 = 0$

Comparing the given equation with $ax^2 + bx + c = 0$

We have, $a = 27$, $b = -10$ and $c = 1$

Hence, discriminant of the equation is

$b^2 - 4ac = (-10)^2 - 4 \times 27 \times 1 = 100 - 108 = -8$

Therefore, the solution of the quadratic equation is

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-\ -10\ \pm \sqrt{-8}}{2 \times 27}$

$= \frac{10 \pm \sqrt{8}i}{2 \times 27}$ [since $\sqrt{-1} = i$]

$= \frac{10 \pm 2\sqrt{2}i}{2 \times 27}$

$= \frac{2\ 5 \pm \sqrt{2}i}{2 \times 27}$

$= \frac{5 \pm \sqrt{2}i}{27}$

$= \frac{5}{27} \pm \frac{\sqrt{2}}{27}i$

Q9. $\mathbf{21x^2 - 28x + 10 = 0}$

A.9. $21x^2 - 28x + 10 = 0$

Comparing the given equation with $ax^2 + bx + c = 0$

We have, $a = 21$, $b = -28$ and $c = 10$

Hence, discriminant of the equation is

$b^2 - 4ac = (-28)^2 - 4 \times 21 \times 10 = 784 - 840 = -56$

Therefore, the solution of the quadratic equation is

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-\ -28\ \pm \sqrt{-56}}{2 \times 21}$

$= \frac{28 \pm \sqrt{56}i}{2 \times 21}$ [since $\sqrt{-1} = i$]

$= \frac{28 \pm 2\sqrt{14}i}{2 \times 21}$

$$= \frac{2\ 14 \pm \sqrt{14}i}{2 \times 21}$$

$$= \frac{14 \pm \sqrt{14}i}{21}$$

$$= \frac{14}{21} \pm \frac{\sqrt{14}}{21}i$$

$$= \frac{2}{3} \pm \frac{\sqrt{14}}{21}i$$

Q10. If $z_1 = 2 - i$, $z_2 = 1 + i$, find $\left|\frac{z_1 + z_2 + 1}{z_1 - z_2 + 1}\right|$

A.10. $z_1 = 2 - i$,$z_2 = 1 + i$

$$\left|\frac{z_1 + z_2 + 1}{z_1 - z_2 + 1}\right|$$

$$= \left|\frac{2 - i + 1 + i + 1}{2 - i - 1 - i + 1}\right|$$

$$= \left|\frac{4}{2 - 2i}\right|$$

$$= \left|\frac{4}{2\ 1 - i}\right|$$

$$= \left|\frac{2}{1i}\right|$$

$$= \left|\frac{2}{1 - i} \times \frac{1 + i}{1 + i}\right|$$ [multiply numerator and denominator by (1 + i)]

$$= \left|\frac{2 + 2i}{1^2 - i^2}\right|$$

$$= \left|\frac{2 + 2i}{1 - \ -1}\right|$$ [since, $i^2 = -1$]

$$= \left|\frac{2 + 2i}{1 + 1}\right|$$

$$= \left|\frac{2\ 1 + i}{2}\right|$$

$= |1 + i|$

$$= \sqrt{1^2 + 1^2}$$ [as $|x + iy| = \sqrt{x^2 + y^2}$]

$$= \sqrt{2}$$

Q11. **If $a + ib = \frac{(x+i)^2}{2x^2+1}$, prove that $a^2 + b^2 = \frac{(x^2+1)^2}{(2x^2+1)^2}$.**

A.11. Let, $z = a + ib$

$$= \frac{(x+i)^2}{2x^2+1}$$

$$= \frac{x^2+i^2+2xi}{2x^2+1} \quad \text{[since, } (a+b)^2 = a^2+b^2+2ab\text{]}$$

$$= \frac{x^2-1+2xi}{2x^2+1} \quad \text{[since, } i^2 = -1\text{]}$$

$$= \frac{x^2-1}{2x^2+1} + \frac{i2x}{2x^2+1}$$

So, $|z|^2 = a^2 + b^2$

$$= \frac{(x^2-1)^2}{(2x^2+1)^2} + \frac{(2x)^2}{(2x^2+1)^2}$$

$$= \frac{(x^2)^2+1^2-2.x^2.1+4x^2}{(2x^2+1)^2} \quad \text{[since, } (a+b)^2 = a^2+b^2+2ab\text{]}$$

$$= \frac{x^4+1-2x^2+4x^2}{(2x^2+1)^2}$$

$$= \frac{x^4+2x^2+1}{(2x^2+1)^2}$$

$$= \frac{(x^2+1)^2}{(2x^2+1)^2} \quad \text{[as, } (a+b)^2 = a^2+b^2+2ab\text{]}$$

Hence proved.

Q12. **Let $z_1 = 2 - i$, $z_2 = -2 + i$. Find**

(i) $\text{RE}\left(\frac{z_1 z_2}{\bar{z}_1}\right)$ **(ii)** $\text{IM}\left(\frac{1}{z_1 \bar{z}_1}\right)$

A.12. $Z_1 = 2 - i$, $z_2 = -2 + i$

$Z_1 z_2 = (2 - i)(-2 + i)$

$= -4 + 2i + 2i - i^2$

$= -4 + 4i + 1$ [since, $i^2 = -1$]

$= -3 + 4i$

$\bar{z}_1 = 2 + i$

i. $\frac{z_1 z_2}{\bar{z}_1} = \frac{-3+4i}{2+i}$

$= \frac{-3+4i}{2+i} \times \frac{2-i}{2-i}$ [multiply denominator and numerator by (2 – i)]

$= \frac{-6+3i+8i-4i^2}{2^2-i^2}$

$= \frac{-6+11i+4}{4- \ -1}$ [since, $i^2 = -1$]

$= \frac{-6+4+11i}{4+1}$

$= \frac{-2+11i}{5}$

$= \frac{-2}{5} + \frac{11}{5}i$

So, Re($\frac{z_1 z_2}{\overline{z_1}}$) = $\frac{-2}{5}$

ii. $\frac{1}{z_1 \overline{z_1}} = \frac{1}{2-i \quad 2+i}$

$= \frac{1}{2^2-i^2}$

$= \frac{1}{4+1}$ [since, $i^2 = -1$]

$= \frac{1}{5} + 0i$

Therefore, $Im\left(\frac{1}{z_1 \overline{z_1}}\right) = 0$

Q13. Find the modulus and argument of the complex number $\frac{1+2i}{1-3i}$.

A.13. Given, z = $\frac{1+2i}{1-3i}$

$= \frac{1+2i}{1-3i} \times \frac{1+3i}{1+3i}$ [multiplying denominator and numerator by (1 + 3i)]

$= \frac{1+3i+2i+6i^2}{1^2-(3i)^2}$

$= \frac{1+5i+6i^2}{1-9i^2}$

$= \frac{1-6+5i}{1+9}$ [since, $i^2 = -1$]

$= \frac{-5+5i}{10}$

$= \frac{5\ -1+i}{10}$

$= \frac{-1+i}{2}$

$= \frac{-1}{2} + \frac{1}{2}i$

Let, $r\cos\theta = \frac{-1}{2}$ and $r\sin\theta = \frac{1}{2}$

Squaring and adding both sides we get,

$r^2(\cos^2\theta + \sin^2\theta) = \left(\frac{-1}{2}\right)^2 + \left(\frac{1}{2}\right)^2$

$\Rightarrow r^2 = \frac{1}{4} + \frac{1}{4}$ (since $\cos^2\theta + \sin^2\theta = 1$)

$\Rightarrow r^2 = \frac{2}{4}$

$\Rightarrow r^2 = \frac{1}{2}$

$\Rightarrow r = \frac{1}{\sqrt{2}}$ (as $r > 0$, $r \neq -\frac{1}{\sqrt{2}}$)

So, $\frac{1}{\sqrt{2}}\cos\theta = \frac{-1}{2}$ and $\frac{1}{\sqrt{2}}\sin\theta = \frac{1}{2}$

$\Rightarrow \cos\theta = \frac{-\sqrt{2}}{2}$ and $\sin\theta = \frac{\sqrt{2}}{2}$

$\Rightarrow \cos\theta = \frac{-\sqrt{2}\sqrt{2}}{2\sqrt{2}}$ and $\sin\theta = \frac{\sqrt{2}\sqrt{2}}{2\sqrt{2}}$

$\Rightarrow \cos\theta = \frac{-2}{2\sqrt{2}}$ and $\sin\theta = \frac{2}{2\sqrt{2}}$

$\Rightarrow \cos\theta = \frac{-1}{\sqrt{2}}$ and $\sin\theta = \frac{1}{\sqrt{2}}$

As, θ lies in 2nd quadrant

$\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

Hence, modules and argument of complex number $\frac{1+2i}{1-3i}$ is $\frac{1}{\sqrt{2}}$ and $\frac{3\pi}{4}$

Q14. Find the real numbers x and y if $(x - iy)(3 + 5i)$ is the conjugate of $-6 - 24i$.

A.14. Let $z = (x - iy)(3 + 5i)$

$= 3x + 5xi - 3yi - 5yi^2$

$= (3x + 5y) + (5x - 3y)i$

Given, $\bar{z} = -6 - 24i$

$\Rightarrow (3x + 5y) - (5x - 3y)i = -6 - 24i$

Equating real and imaginary part,

$3x + 5y = –6$ ----------- (1)

$5x – 3y = 24$ ----------- (2)

Multiplying (1) by 3 and (2) by 5 and adding them, we get

$9x + 15y + 25x – 15y = –18 + 120$

$=> 34x = 102$

$=> x = 102/34 = 3$

Putting $x = 3$ in (1) we get,

$3 \times 3 + 5y = –6$

$=> 9 + 5y = –6$

$=> 5y = –6 – 9$

$=> 5y = –15$

$=> y = –15/5 = –3$

Hence, the values of x and y are 3 and –3 respectively.

Q15. Find the modulus of $\frac{1+i}{1-i} - \frac{1-i}{1+i}$.

A.15. $\frac{1+i}{1-i} - \frac{1-i}{1+i}$

$= \frac{(1+i)^2 - (1-i)^2}{(1-i)(1+i)}$

$= \frac{1^2 + i^2 + 2.1.i - (1^2 + i^2 - 2.1.i)}{1^2 - i^2}$ [Since, $(a + b)^2 = a^2 + b^2 + 2ab$;

$(a – b)^2 = a^2 + b^2 – 2ab$;

$a^2 – b^2 = (a + b)(a – b)$]

$= \frac{1-1+2i-1+1+2i}{1+1}$ [Since, $i^2 = –1$]

$= \frac{4i}{2}$

$= 2i$

Hence, $\left|\frac{1+i}{1-i} - \frac{1-i}{1+i}\right| = \sqrt{2^2} = 2$

Q16. If $(x + iy)^3 = u + iv$, then show that $\frac{u}{x} + \frac{v}{y} = 4(x^2 - y^2)$.

A.16. $(x + iy)^3 = u + iv$

$=> x^3 + (iy)^3 + 3.x.iy(x + iy) = u + iv$ [since, $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$]

$=> x^3 – iy^3 + 3x^2yi + 3xy^2i^2 = u + iv$

$=> x^3 – iy^3 + 3x^2yi – 3xy^2 = u + iv$ [since, i^2 = -1]

$=> (x^3 – 3xy^2) + i(3x^2y – y^3) = u + iv$

Equating real and imaginary part we get,

$u = x^3 – 3xy^2$ and $v = 3x^2y – y^3$

Now, $\frac{u}{x} + \frac{v}{y}$

$$= \frac{x^3 - 3xy^2}{x} + \frac{3x^2y - y^3}{y}$$

$$= \frac{x(x^2 - 3y^2)}{x} + \frac{y(3x^2 - y^2)}{y}$$

$= x^2 - 3y^2 + 3x^2 - y^2$

$= 4x^2 - 4y^2$

$= 4(x^2 - y^2)$

Hence proved.

Q17. If α and β are different complex numbers with $|\beta| = 1$ then find $\left|\frac{\beta - \alpha}{1 - \bar{\alpha}\beta}\right|$.

A.17. Let, $\alpha = a + ib$

$\beta = c + id$

Given, $|\beta| = 1$

$\Rightarrow \sqrt{c^2 + d^2} = 1$

$\Rightarrow c^2 + d^2 = 1$ ---------(1)

By question,

$$\left|\frac{\beta - \alpha}{1 - \bar{\alpha}\beta}\right| = \left|\frac{(c + id) - (a + ib)}{1 - (a - ib)(c + id)}\right|$$

$$= \left|\frac{c + id - a - ib}{1 - (ac + iad - ibc - i^2bd)}\right|$$

$$= \left|\frac{(c - a) + i(d - b)}{1 - [ac + bd + i(ad - bc)]}\right| \quad [\text{since, } i^2 = -1]$$

$$= \left|\frac{(c - a) + i(d - b)}{1 - ac - bd - i(ad - bc)}\right|$$

$$= \left|\frac{(c - a) + i(d - b)}{(1 - ac - bd) + i(bc - ad)}\right|$$

We know that $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, hence,

$$\left|\frac{\beta - \alpha}{1 - \bar{\alpha}\beta}\right| = \frac{\sqrt{(c - a)^2 + (d - b)^2}}{\sqrt{(1 - ac - bd)^2 + (bc - ad)^2}}$$

$$= \frac{\sqrt{c^2 + a^2 - 2ac + d^2 + b^2 - 2db}}{\sqrt{1 + a^2c^2 + b^2d^2 - 2ac + 2abcd - 2bd + b^2c^2 + a^2d^2 - 2abcd}}$$

[since, $(a - b)^2 = a^2 + b^2 - 2ab$; $(a - b - c)^2 = a^2 + b^2 + c^2 - 2ab + 2bc - 2ca$]

$$= \frac{\sqrt{(c^2 + d^2) + a^2 + b^2 - 2ac - 2db}}{\sqrt{1 + a^2(c^2 + d^2) + b^2(d^2 + c^2) - 2ac - 2bd}}$$

= $\dfrac{\sqrt{1+a^2+b^2-2ac-2db}}{\sqrt{1+a^2+b^2-2ac-2bd}}$ [since, $(c^2 + d^2) = 1$, from (1)]

= 1

Therefore, $\left|\dfrac{\beta-\alpha}{1-\bar{\alpha}\beta}\right| = 1$

Q18. Find the number of non-zero integral solutions of the equation $|1-i|^x = 2^x$.

A.18. $|1 - i|^x = 2^x$

=> $(\sqrt{1+1})^x = 2^x$

=> $\sqrt{2}^{\,x} = 2^x$

=> $2^{x/2} = 2^x$

So, $\dfrac{x}{2} = 2x$

=> $x = 2x$

=> $2x - x = 0$

=> $x = 0$

So, the only solution of the given equation is 0.

Hence, there is no non – zero integral solution of the given equation.

Q19. If $(a + ib)(c + id)(e + if)(g + ih) = A + iB$, then show that $(a^2 + b^2)(c^2 + d^2)(e^2 + f^2)(g^2 + h^2) = A^2 + B^2$

A.19. Given,

$(a + ib)(c + id)(e + if)(g + ih) = A + iB$

We know that,

$|z_1 z_2| = |z_1||z_2|$ hence we can write

$|A + iB| = |a + ib||c + id||e + if||g + ih|$

=> $\sqrt{A^2+B^2} = \sqrt{a^2+b^2}\sqrt{c^2+d^2}\sqrt{e^2+f^2}\sqrt{g^2+h^2}$

=> $A^2 + B^2 = (a^2 + b^2)(c^2 + d^2)(e^2 + f^2)(g^2 + h^2)$

Hence proved.

Q20. If $\left(\dfrac{1+i}{1-i}\right)^m = 1$, then find the least positive integral value of *m*.

A.20. We have,

$\left(\dfrac{1+i}{1-i}\right)^m = 1$

=> $\left(\dfrac{1+i}{1-i}\times\dfrac{1+i}{1+i}\right)^m = 1$ [multiply denominator and numerator of LHS by $(1 + i)$]

=> $\left(\dfrac{1+i+i+i^2}{1^2-i^2}\right)^m = 1$ [since, $(a - b)(a + b) = a^2 - b^2$]

=> $\left(\dfrac{1+2i-1}{1+1}\right)^m = 1$ [since, $i^2 = -1$]

=> $\left(\frac{2i}{2}\right)^m = 1$

=> $i^m = 1$

=> $i^m = i^{4k}$ [since, $i^{4k} = 1$]

So, $m = 4k$ where k = integer

Therefore, least positive integral value of m is,

$m = 4 \times 1$

$m = 4$