MISCELLANEOUS EXERCISE

Prove that:

Question 1:

$$2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}=0$$

Solution:

$$LHS=2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}$$

$$=2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+2\cos\left(\frac{\frac{3\pi}{13}+\frac{5\pi}{13}}{2}\right)\cos\left(\frac{\frac{3\pi}{13}-\frac{5\pi}{13}}{2}\right)\quad\left[\because \cos A+\cos B=2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\right]$$

$$=2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+2\cos\frac{4\pi}{13}\cos\left(-\frac{\pi}{13}\right)$$

$$=2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+2\cos\frac{4\pi}{13}\cos\frac{\pi}{13}$$

$$=2\cos\frac{\pi}{13}\left[\cos\frac{9\pi}{13}+\cos\frac{4\pi}{13}\right]$$

$$=2\cos\frac{\pi}{13}\left[2\cos\left(\frac{\frac{9\pi}{13}+\frac{4\pi}{13}}{2}\right)\cos\left(\frac{\frac{9\pi}{13}-\frac{4\pi}{13}}{2}\right)\right]\quad\left[\because \cos A+\cos B=2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\right]$$

$$=2\cos\frac{\pi}{13}\left[2\cos\frac{\pi}{2}\cos\frac{5\pi}{26}\right]$$

$$=2\cos\frac{\pi}{13}\times 2\times 0\times\cos\frac{5\pi}{26}\quad\left[\because \cos\frac{\pi}{2}=0\right]$$

$$=0$$

$$=RHS$$

Question 2:

$$(\sin 3x+\sin x)\sin x+(\cos 3x-\cos x)\cos x=0$$

Solution:

$$LHS = (\sin 3x + \sin x)\sin x + (\cos 3x - \cos x)\cos x$$
$$= \sin 3x \sin x + \sin^2 x + \cos 3x \cos x - \cos^2 x$$
$$= \cos 3x \cos x + \sin 3x \sin x - (\cos^2 x - \sin^2 x)$$
$$= \cos(3x - x) - \cos 2x \qquad \left[\begin{array}{l} \because \cos(A-B) = \cos A \cos B + \sin A \sin B \\ \& \cos 2A = \cos^2 A - \sin^2 A \end{array}\right]$$
$$= \cos 2x - \cos 2x$$
$$= 0$$
$$= RHS$$

Question 3:

$$(\cos x + \cos y)^2 + (\sin x - \sin y)^2 = 4\cos^2 \frac{x+y}{2}$$

Solution:

$$LHS = (\cos x + \cos y)^2 + (\sin x - \sin y)^2$$
$$= \cos^2 x + \cos^2 y + 2\cos x \cos y + \sin^2 x + \sin^2 y - 2\sin x \sin y \qquad \left[\begin{array}{l} \because (a+b)^2 = a^2 + b^2 + 2ab \\ \& (a-b)^2 = a^2 + b^2 - 2ab \end{array}\right]$$
$$= (\cos^2 x + \sin^2 x) + (\cos^2 y + \sin^2 y) + 2(\cos x \cos y - \sin x \sin y)$$
$$= 1 + 1 + 2\cos(x+y) \qquad \left[\begin{array}{l} \because (\cos^2 A + \sin^2 A) = 1 \\ \& \cos(A+B) = \cos A \cos B - \sin A \sin B \end{array}\right]$$
$$= 2 + 2\cos(x+y)$$
$$= 2\left[1 + \cos 2\left(\frac{x+y}{2}\right)\right]$$
$$= 2\left[1 + 2\cos^2\left(\frac{x+y}{2}\right) - 1\right] \qquad \left[\because \cos 2A = 2\cos^2 A - 1\right]$$
$$= 4\cos^2 \frac{x+y}{2}$$
$$= RHS$$

Question 4:

$$(\cos x - \cos y)^2 + (\sin x - \sin y)^2 = 4\sin^2 \frac{x-y}{2}$$

Solution:

$$LHS = (\cos x - \cos y)^2 + (\sin x - \sin y)^2$$

$$= \cos^2 x + \cos^2 y - 2\cos x \cos y + \sin^2 x + \sin^2 y - 2\sin x \sin y \quad \left[\because (a+b)^2 = a^2 + b^2 + 2ab \;\&\; (a-b)^2 = a^2 + b^2 - 2ab\right]$$

$$= (\cos^2 x + \sin^2 x) + (\cos^2 y + \sin^2 y) - 2(\cos x \cos y + \sin x \sin y)$$

$$= 1 + 1 - 2\cos(x-y) \quad \left[\because (\cos^2 A + \sin^2 A) = 1 \;\&\; \cos(A-B) = \cos A \cos B + \sin A \sin B\right]$$

$$= 2 - 2\cos(x-y)$$

$$= 2\left[1 - \cos 2\left(\frac{x-y}{2}\right)\right]$$

$$= 2\left[1 - \left\{1 - 2\sin^2\left(\frac{x-y}{2}\right)\right\}\right] \quad \left[\because \cos 2A = 2\cos^2 A - 1\right]$$

$$= 2\left[1 - 1 + 2\sin^2\left(\frac{x-y}{2}\right)\right]$$

$$= 4\sin^2 \frac{x-y}{2}$$

$$= RHS$$

Question 5:

$\sin x + \sin 3x + \sin 5x + \sin 7x = 4\cos x \cos 2x \sin 4x$

Solution:

$$LHS = \sin x + \sin 3x + \sin 5x + \sin 7x$$

$$= (\sin 5x + \sin x) + (\sin 7x + \sin 3x)$$

$$= \left[2\sin\left(\frac{5x+x}{2}\right)\cos\left(\frac{5x-x}{2}\right)\right] + \left[2\sin\left(\frac{7x+3x}{2}\right)\cos\left(\frac{7x-3x}{2}\right)\right]$$

$$\left[\because \sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\right]$$

$$= 2\sin 3x \cos 2x + 2\sin 5x \cos 2x$$

$$= 2\cos 2x(\sin 5x + \sin 3x)$$

$$= 2\cos 2x\left[2\sin\left(\frac{5x+3x}{2}\right)\cos\left(\frac{5x-3x}{2}\right)\right] \quad \left[\because \sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\right]$$

$$= 2\cos 2x[2\sin 4x \cos x]$$

$$= 4\cos x \cos 2x \sin 4x$$

$$= RHS$$

Question 6:

$$\frac{(\sin 7x+\sin 5x)+(\sin 9x+\sin 3x)}{(\cos 7x+\cos 5x)+(\cos 9x+\cos 3x)}=\tan 6x$$

Solution:

$$LHS=\frac{(\sin 7x+\sin 5x)+(\sin 9x+\sin 3x)}{(\cos 7x+\cos 5x)+(\cos 9x+\cos 3x)}$$

$$=\frac{2\sin\left(\frac{7x+5x}{2}\right)\cos\left(\frac{7x-5x}{2}\right)+2\sin\left(\frac{9x+3x}{2}\right)\cos\left(\frac{9x-3x}{2}\right)}{2\cos\left(\frac{7x+5x}{2}\right)\cos\left(\frac{7x-5x}{2}\right)+2\cos\left(\frac{9x+3x}{2}\right)\cos\left(\frac{9x-3x}{2}\right)}$$

$$\left[\begin{array}{l}\because\ \sin A+\sin B=2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\\ \&\ \cos A+\cos B=2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\end{array}\right]$$

$$=\frac{2\sin 6x\cos x+2\sin 6x\cos 3x}{2\cos 6x\cos x+2\cos 6x\cos 3x}$$

$$=\frac{2\sin 6x(\cos x+\cos 3x)}{2\cos 6x(\cos x+\cos 3x)}$$

$$=\frac{\sin 6x}{\cos 6x}$$

$$=\tan 6x$$

$$=RHS$$

Question 7:

$$\sin 3x+\sin 2x-\sin x=4\sin x\cos\frac{x}{2}\cos\frac{3x}{2}$$

Solution:

$$LHS = \sin 3x + \sin 2x - \sin x$$

$$= \sin 3x + (\sin 2x - \sin x)$$

$$= \sin 2\left(\frac{3x}{2}\right) + \left[2\cos\left(\frac{2x+x}{2}\right)\sin\left(\frac{2x-x}{2}\right)\right] \qquad \left[\because \sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\right]$$

$$= 2\sin\frac{3x}{2}\cos\frac{3x}{2} + 2\cos\frac{3x}{2}\sin\frac{x}{2} \qquad [\because \sin 2A = 2\sin A\cos A]$$

$$= 2\cos\frac{3x}{2}\left(\sin\frac{3x}{2} + \sin\frac{x}{2}\right)$$

$$= 2\cos\frac{3x}{2}\left[2\sin\left(\frac{\frac{3x}{2}+\frac{x}{2}}{2}\right)\cos\left(\frac{\frac{3x}{2}-\frac{x}{2}}{2}\right)\right] \qquad \left[\because \sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\right]$$

$$= 2\cos\frac{3x}{2}\left[2\sin x\cos\frac{x}{2}\right]$$

$$= 4\sin x\cos\frac{x}{2}\cos\frac{3x}{2}$$

$$= RHS$$

Find $\sin\frac{x}{2}$, $\cos\frac{x}{2}$ and $\tan\frac{x}{2}$ in each of the following:

Question 8:

$\tan x = -\frac{4}{3}$, x in quadrant II.

Solution:

Since x lies in quadrant II

$$\frac{\pi}{2} < x < \pi$$

Therefore,

$$\frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$$

Hence, $\sin\frac{x}{2}$, $\cos\frac{x}{2}$ and $\tan\frac{x}{2}$ lie in quadrant I and all are positive.

It is given that $\tan x = -\frac{4}{3}$

$$\sec^2 x = 1 + \tan^2 x$$
$$= 1 + \left(-\frac{4}{3}\right)^2$$
$$= 1 + \frac{16}{9}$$
$$= \frac{25}{9}$$
$$\sec x = \pm\sqrt{\frac{25}{9}}$$
$$\frac{1}{\cos x} = \pm\frac{5}{3}$$
$$\cos x = \pm\frac{3}{5}$$

As x is in quadrant II, $\cos x$ is negative.

$$\cos x = -\frac{3}{5}$$
$$\cos 2\left(\frac{x}{2}\right) = -\frac{3}{5}$$
$$2\cos^2\frac{x}{2} - 1 = -\frac{3}{5}$$
$$2\cos^2\frac{x}{2} = 1 - \frac{3}{5}$$
$$\cos^2\frac{x}{2} = \frac{2}{5}\times\frac{1}{2}$$
$$\cos^2\frac{x}{2} = \frac{1}{5}$$
$$\cos\frac{x}{2} = \pm\sqrt{\frac{1}{5}}$$

Since, $\cos\frac{x}{2}$ lies in quadrant I and positive, $\cos\frac{x}{2} = \frac{1}{\sqrt{5}}$

Now,

$$\sin^2\frac{x}{2}=1-\cos^2\frac{x}{2} \qquad [\because \sin^2 A+\cos^2 A=1]$$

$$=1-\left(\frac{1}{\sqrt{5}}\right)^2$$

$$=1-\frac{1}{5}$$

$$=\frac{4}{5}$$

$$\sin\frac{x}{2}=\pm\sqrt{\frac{4}{5}}$$

$$\sin\frac{x}{2}=\pm\frac{2}{\sqrt{5}}$$

Since, $\sin\frac{x}{2}$ lies in quadrant I and positive, $\sin\frac{x}{2}=\frac{2}{\sqrt{5}}$

Now,

$$\tan\frac{x}{2}=\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}$$

$$=\frac{\left(\frac{2}{\sqrt{5}}\right)}{\left(\frac{1}{\sqrt{5}}\right)}$$

$$=\frac{2}{\sqrt{5}}\times\frac{\sqrt{5}}{1}$$

$$=2$$

Therefore, $\sin\frac{x}{2}=\frac{2}{\sqrt{5}}$, $\cos\frac{x}{2}=\frac{1}{\sqrt{5}}$ and $\tan\frac{x}{2}=2$.

Question 9:

$\cos x=-\frac{1}{3}, x$ in quadrant III.

Solution:

Since x lies in quadrant III

$$\pi<x<\frac{3\pi}{2}$$

Therefore,

$$\frac{\pi}{2}<\frac{x}{2}<\frac{3\pi}{4}$$

Hence, , $\cos\frac{x}{2}$ and $\tan\frac{x}{2}$ are negative while $\sin\frac{x}{2}$ is positive as all lie in quadrant II.

It is given that $\cos x = -\frac{1}{3}$

$$\cos 2\left(\frac{x}{2}\right) = -\frac{1}{3}$$

$$2\cos^2\frac{x}{2} - 1 = -\frac{1}{3}$$

$$2\cos^2\frac{x}{2} = 1 - \frac{1}{3}$$

$$\cos^2\frac{x}{2} = \frac{2}{3} \times \frac{1}{2}$$

$$\cos^2\frac{x}{2} = \frac{1}{3}$$

$$\cos\frac{x}{2} = \pm\sqrt{\frac{1}{3}}$$

Since, $\cos\frac{x}{2}$ is negative

So,

$$\cos\frac{x}{2} = -\frac{1}{\sqrt{3}}$$

$$= -\frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= -\frac{\sqrt{3}}{3}$$

Now,

$$\sin^2\frac{x}{2} = 1 - \cos^2\frac{x}{2} \qquad \left[\because \sin^2 A + \cos^2 A = 1\right]$$

$$= 1 - \left(-\frac{\sqrt{3}}{3}\right)^2$$

$$= 1 - \frac{1}{3}$$

$$= \frac{2}{3}$$

$$\sin\frac{x}{2} = \pm\sqrt{\frac{2}{3}}$$

Since, $\sin\frac{x}{2}$ positive,

So,

$$\sin\frac{x}{2}=\sqrt{\frac{2}{3}}$$
$$=\frac{\sqrt{2}}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}$$
$$=\frac{\sqrt{6}}{3}$$

Now,

$$\tan\frac{x}{2}=\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}$$
$$=\frac{\frac{\sqrt{6}}{3}}{\left(-\frac{\sqrt{3}}{3}\right)}$$
$$=\frac{\sqrt{6}}{3}\times\left(-\frac{3}{\sqrt{3}}\right)$$
$$=-\sqrt{2}$$

Therefore, $\sin\frac{x}{2}=\frac{\sqrt{6}}{3}$, $\cos\frac{x}{2}=-\frac{\sqrt{3}}{3}$ and $\tan\frac{x}{2}=-\sqrt{2}$.

Question 10:

$\sin x=\frac{1}{4}, x$ in quadrant II.

Solution:

Since x lies in quadrant II

$$\frac{\pi}{2}<x<\pi$$

Therefore,

$$\frac{\pi}{4}<\frac{x}{2}<\frac{\pi}{2}$$

Hence, $\sin\frac{x}{2}$, $\cos\frac{x}{2}$ and $\tan\frac{x}{2}$ lie in quadrant I and all are positive.

It is given that $\sin x=\frac{1}{4}$

Therefore,

$$\cos^2 x = 1 - \sin^2 x \qquad [\because \sin^2 A + \cos^2 A = 1]$$

$$= 1 - \left(\frac{1}{4}\right)^2$$

$$= 1 - \frac{1}{16}$$

$$= \frac{15}{16}$$

$$\cos x = \pm\sqrt{\frac{15}{16}}$$

Since, $\cos x$ lies in quadrant II and negative

So,

$$\cos x = -\sqrt{\frac{15}{16}}$$

$$\cos 2\left(\frac{x}{2}\right) = -\frac{\sqrt{15}}{4}$$

$$2\cos^2\frac{x}{2} - 1 = -\frac{\sqrt{15}}{4}$$

$$2\cos^2\frac{x}{2} = 1 - \frac{\sqrt{15}}{4}$$

$$2\cos^2\frac{x}{2} = \frac{4-\sqrt{15}}{4}$$

$$\cos^2\frac{x}{2} = \frac{4-\sqrt{15}}{8}$$

$$\cos\frac{x}{2} = \pm\sqrt{\frac{4-\sqrt{15}}{8}}$$

Since, $\cos\frac{x}{2}$ lies in quadrant I and is positive.

So,

$$\cos\frac{x}{2} = \sqrt{\frac{4-\sqrt{15}}{8}}$$

$$= \sqrt{\frac{4-\sqrt{15}}{8} \times \frac{2}{2}}$$

$$= \frac{\sqrt{8-2\sqrt{15}}}{4}$$

Now,

$$\sin^2\frac{x}{2} = 1 - \cos^2\frac{x}{2} \qquad [\because \sin^2 A + \cos^2 A = 1]$$

$$= 1 - \left(\frac{\sqrt{8 - 2\sqrt{15}}}{4}\right)^2$$

$$= 1 - \frac{8 - 2\sqrt{15}}{16}$$

$$= \frac{16 - 8 + 2\sqrt{15}}{16}$$

$$= \frac{8 + 2\sqrt{15}}{16}$$

$$\sin\frac{x}{2} = \pm\sqrt{\frac{8 + 2\sqrt{15}}{16}}$$

Since, $\sin\frac{x}{2}$ positive,

So,

$$\sin\frac{x}{2} = \frac{\sqrt{8 + 2\sqrt{15}}}{4}$$

Now,

$$\tan\frac{x}{2}=\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}$$

$$=\frac{\left(\frac{\sqrt{8+2\sqrt{15}}}{4}\right)}{\left(\frac{\sqrt{8-2\sqrt{15}}}{4}\right)}$$

$$=\left(\frac{\sqrt{8+2\sqrt{15}}}{4}\right)\times\left(\frac{4}{\sqrt{8-2\sqrt{15}}}\right)$$

$$=\sqrt{\frac{2\left(4+\sqrt{15}\right)}{2\left(4-\sqrt{15}\right)}}$$

$$=\sqrt{\frac{4+\sqrt{15}}{4-\sqrt{15}}\times\frac{4+\sqrt{15}}{4+\sqrt{15}}}$$

$$=\sqrt{\frac{\left(4+\sqrt{15}\right)^2}{16-15}}$$

$$=4+\sqrt{15}$$

Therefore, $\sin\frac{x}{2}=\frac{\sqrt{8+2\sqrt{15}}}{4}$, $\cos\frac{x}{2}=\frac{\sqrt{8-2\sqrt{15}}}{4}$ and $\tan\frac{x}{2}=4+\sqrt{15}$.