MISCELLANEOUS EXERCISE

Question 1:

Show that the sum of $(m+n)^{th}$ and $(m-n)^{th}$ terms of an A.P is equal to twice the m^{th} term.

Solution:

Let a and d be the first term and common difference of the A.P respectively.

It is known that the k^{th} term of an A.P. is given by $a_k = a+(k-1)d$

Therefore,

$$a_{m+n} = a+(m+n-1)d$$
$$a_{m-n} = a+(m-n-1)d$$
$$a_m = a+(m-1)d$$

Hence,

$$\begin{aligned} a_{m+n}+a_{m-n} &= a+(m+n-1)d+a+(m-n-1)d \\ &= 2a+(m+n-1+m-n-1)d \\ &= 2a+(2m-2)d \\ &= 2a+2(m-1)d \\ &= 2\left[a+(m-1)d\right] \\ &= 2a_m \end{aligned}$$

Thus, the sum of $(m+n)^{th}$ and $(m-n)^{th}$ terms of an A.P is equal to twice the m^{th} term.

Question 2:

Let the sum of three numbers in A.P is 24 and their product is 440. Find the numbers.

Solution:

Let the three numbers in A.P be $(a-d), a$ and $(a+d)$.

According to the given information,

$$(a-d)+(a)+(a-d)=24$$
$$\Rightarrow 3a=24$$
$$\Rightarrow a=8 \qquad \ldots(1)$$

And

$$(a-d)(a)(a-d)=440$$
$$\Rightarrow (8-d)(8)(8+d)=440 \qquad \ldots[\text{from } (1)]$$
$$\Rightarrow (8-d)(8+d)=55$$
$$\Rightarrow 64-d^2=55$$
$$\Rightarrow d^2=9$$
$$\Rightarrow d=\pm 3$$

Therefore,
when $d=3$, the numbers are $5,8$ and 11
when $d=-3$, the numbers are 11, 8 and 5.

Thus, the three numbers are 5,8 and 11.

Question 3:

Let the sum of $n, 2n, 3n$ terms of an A.P be S_1, S_2, S_3 respectively. Show that $S_3 = 3(S_2 - S_1)$.

Solution:

Let a and d be the first term and common difference of the A.P respectively.
Therefore,

$$S_1 = \frac{n}{2}[2a+(n-1)d] \qquad \ldots(1)$$
$$S_2 = \frac{2n}{2}[2a+(2n-1)d]$$
$$= n[2a+(2n-1)d] \qquad \ldots(2)$$
$$S_3 = \frac{3n}{2}[2a+(3n-1)d] \qquad \ldots(3)$$

By subtracting (1) and (2), we obtain

$$S_2 - S_1 = n\left[2a+(2n-1)d\right] - \frac{n}{2}\left[2a+(n-1)d\right]$$
$$= n\left[\frac{4a+4nd-2d-2a-nd+d}{2}\right]$$
$$= n\left[\frac{2a+3nd-d}{2}\right]$$
$$= \frac{n}{2}\left[2a+(3n-1)d\right]$$
$$3(S_2 - S_1) = \frac{3n}{2}\left[2a+(3n-1)d\right]$$
$$= S_3 \qquad \left[\text{from } (3)\right]$$

Hence, $S_3 = 3(S_2 - S_1)$ proved.

Question 4:

Find the sum of all numbers between 200 and 400 which are divisible by 7.

Solution:

The numbers lying between 200 and 400 which are divisible by 7 are $203, 210, 217, \ldots 399$

Here, $a = 203, d = 7$ and $a_n = 399$

Therefore,

$$a_n = a + (n-1)d$$
$$399 = 203 + (n-1)7$$
$$(n-1)7 = 196$$
$$n - 1 = 28$$
$$\Rightarrow n = 29$$

Hence,

$$S_{29} = \frac{29}{2}(203+399)$$
$$= \frac{29}{2}(602)$$
$$= 29 \times 301$$
$$= 8729$$

Thus, the required sum is 8729.

Question 5:

Find the sum of all integers from 1 and 100 that are divisible by 2 or 5.

Solution:

The integers from 1 and 100 that are divisible by 2 are $2,4,6,\ldots 100$

This forms an A.P with both the first term and common difference equal to 2.

Therefore,

$$a_n = a+(n-1)d$$
$$100 = 2+(n-1)2$$
$$100 = 2+2n-2$$
$$\Rightarrow 2n = 100$$
$$\Rightarrow n = 50$$

Therefore,

$$2+4+6+\ldots+100 = \frac{50}{2}[2(2)+(50-1)(2)]$$
$$= \frac{50}{2}[4+98]$$
$$= 25\times 102$$
$$= 2550$$

The integers from 1 to 100 that are divisible by 5 are $5,10,15,\ldots 100$

This forms an A.P with both the first term and common difference equal to 5.

Therefore,

$$a_n = a+(n-1)d$$
$$100 = 5+(n-1)5$$
$$100 = 5+5n-5$$
$$\Rightarrow 5n = 100$$
$$\Rightarrow n = 20$$

Therefore,

$$5+10+15+\ldots+100 = \frac{20}{2}[2(5)+(20-1)(5)]$$
$$= 10[10+(19)5]$$
$$= 10[10+95]$$
$$= 10\times 105$$
$$= 1050$$

The integers, which are divisible by both 2 and 5 are $10,20,30,\ldots 100$

This forms an A.P with both the first term and common difference equal to 10.

Therefore,

$$100 = 10 + (n-1)10$$
$$\Rightarrow 10n = 100$$
$$\Rightarrow n = 10$$

Hence,

$$10 + 20 + \ldots + 100 = \frac{10}{2}[2(10) + (10-1)(10)]$$
$$= 5[20 + 90]$$
$$= 5 \times 110$$
$$= 550$$

Therefore, required sum $= 2550 + 1050 - 550 = 3050$

Thus, the sum of all integers from 1 and 100 that are divisible by 2 or 5 is 3050.

Question 6:

Find the sum of all two-digit numbers which when divided by 4, yields 1 as remainder.

Solution:

The two-digit numbers which when divided by 4, yields 1 as remainder are $13, 17, 21, \ldots 97$.

This forms an A.P with first term 13 and common difference 4.

Let n be the number of terms of the A.P.

It is known that the n^{th} term of an A.P. is given by $a^n = a + (n-1)d$

Therefore,

$$97 = 13 + (n-1)(4)$$
$$4(n-1) = 97 - 13$$
$$n - 1 = \frac{84}{4}$$
$$\Rightarrow n = 22$$

Sum of n terms of an A.P, $S_n = \frac{n}{2}[2a + (n-1)d]$

Therefore,

$$\begin{aligned}S_{22} &= \frac{22}{2}\left[2(13)+(22-1)(4)\right]\\ &= 11[26+84]\\ &= 11\times 110\\ &= 1210\end{aligned}$$

Thus, the required sum is 1210.

Question 7:

If f is a fraction satisfying $f(x+y)=f(x)\cdot f(y)$ for all $x, y \in N$, such that $f(1)=3$ and $\sum_{x=1}^{n} f(x)=120$, find the value of n.

Solution:

$f(x+y)=f(x)\cdot f(y)$ for all $x, y \in N$ and $f(1)=3$

Taking $x=y=1$ in (1), we obtain

$$\begin{aligned}f(x+y) &= f(x)\cdot f(y)\\ f(1+1) &= f(1)f(1)\\ f(2) &= 3\times 3\\ &= 9\end{aligned}$$

Similarly,

$$\begin{aligned}f(x+y) &= f(x)\cdot f(y)\\ f(1+2) &= f(1)f(2)\\ f(3) &= 3\times 9\\ &= 27\end{aligned}$$

Also,

$$\begin{aligned}f(x+y) &= f(x)\cdot f(y)\\ f(1+3) &= f(1)f(3)\\ f(4) &= 3\times 27\\ &= 81\end{aligned}$$

Therefore, $f(1), f(2), f(3)$ i.e., $3,9,27$ forms a G.P. with both the first term and common ratio equal to 3.

It is known that $S_n = \dfrac{a(r^n-1)}{r-1}$

It is given that $\sum_{k=1}^{n} f(x) = 120$

Therefore,

$$\Rightarrow 120 = \frac{3(3^n - 1)}{3-1}$$
$$\Rightarrow 120 = \frac{3}{2}(3^n - 1)$$
$$\Rightarrow 3^n - 1 = 80$$
$$\Rightarrow 3^n = 81$$
$$\Rightarrow 3^n = 3^4$$
$$\Rightarrow n = 4$$

Thus, the value of $n = 4$.

Question 8:

The sum of some terms of G.P is 315 whose first term and common ratio are 5 and 2 respectively. Find the last term and the number of terms.

Solution:

Let the sum of n terms of the G.P be 315.

It is known that $S_n = \frac{a(r^n - 1)}{r-1}$

It is given that $a = 5$ and common ratio $r = 2$

Hence,

$$315 = \frac{5(2^n - 1)}{2-1}$$
$$\Rightarrow 2^n - 1 = 63$$
$$\Rightarrow 2^n = 64 = (2)^6$$
$$\Rightarrow n = 6$$

Therefore, last term of the G.P. is 6th term

$$a_6 = ar^{6-1}$$
$$= ar^5$$
$$= 5 \times (2)^5$$
$$= 5 \times 32$$
$$= 160$$

Thus, the last term of the G.P is 160.

Question 9:

The first term of a G.P is 1. The sum of the third term and fifth term is 90. Find the common ratio of G.P.

Solution:

Let a and r be the first term and common ratio of the G.P respectively.

Hence,

$$a = 1$$
$$a_3 = ar^2 = r^2$$
$$a_5 = ar^4 = r^4$$

Therefore,

$$r^2 + r^4 = 90$$
$$r^2 + r^4 - 90 = 0$$
$$r^2 = \frac{-1+\sqrt{1+360}}{2}$$
$$= \frac{-1\pm\sqrt{361}}{2}$$
$$= \frac{-1\pm 19}{2}$$
$$= -10 \text{ or } 9$$
$$r^2 = 9 \qquad [\text{Taking real roots}]$$
$$r = \pm 3$$

Thus, the common ratio of the G.P is ± 3.

Question 10:

The sum of three numbers in G.P is 56. If we subtract $1, 7, 21$ from these numbers in that order, we obtain an A.P. Find the numbers.

Solution:

Let the three numbers in G.P. be a, ar and ar^2.

From the given condition,

$$a + ar + ar^2 = 56$$
$$\Rightarrow a(1 + r + r^2) = 56 \qquad \ldots(1)$$

Now, $(a-1), (ar-7), (ar^2-21)$ forms an A.P.

Therefore,

$$(ar-7)-(a-1)=(ar^2-21)-(ar-7)$$
$$\Rightarrow ar-a-6=ar^2-ar-14$$
$$\Rightarrow ar^2-2ar+a=8$$
$$\Rightarrow ar^2-ar-ar+a=8$$
$$\Rightarrow a(r^2+1-2r)=8$$
$$\Rightarrow 7a(r^2-2r+1)=56 \qquad \ldots(2)$$

From (1) and (2), we get

$$\Rightarrow 7(r^2-2r+1)=(1+r+r^2)$$
$$\Rightarrow 7r^2-14r+7-1-r-r^2=0$$
$$\Rightarrow 6r^2-15r+6=0$$
$$\Rightarrow 2r^2-5r+2=0$$
$$\Rightarrow 2r^2-4r-r+2=0$$
$$\Rightarrow 2r(r-2)-1(r-2)=0$$
$$\Rightarrow (2r-1)(r-2)=0$$
$$\Rightarrow r=\frac{1}{2},2$$

<u>Case I</u>:

Substituting $r=2$ in (1), we obtain

$$a(1+2+2^2)=56$$
$$\Rightarrow 7a=56$$
$$\Rightarrow a=8$$

Hence, the three numbers are in G.P are 8, 16 and 32.

<u>Case II</u>:

Substituting $r=\frac{1}{2}$ in (1), we obtain

$$a\left(1+\frac{1}{2}+\left(\frac{1}{2}\right)^2\right)=56$$
$$\Rightarrow \frac{7}{4}a=56$$
$$\Rightarrow a=32$$

Hence, the three numbers are in G.P are 32, 16 and 8.

Thus, the three required numbers are 8, 16 and 32.

Question 11:
A G.P consists of an even number of terms. If the sum of all terms is 5 times the sum of terms occupying odd places, then find its common ratio.

Solution:

Let the G.P be $T_1, T_2, T_3, T_4, \ldots T_{2n}$.
According to the question,

$$\Rightarrow T_1 + T_2 + T_3 + T_4 + \ldots + T_{2n} = 5[T_1 + T_3 + T_5 + \ldots + T_{2n-1}]$$
$$\Rightarrow T_1 + T_2 + T_3 + T_4 + \ldots + T_{2n} - 5[T_1 + T_3 + T_5 + \ldots + T_{2n-1}] = 0$$
$$\Rightarrow T_2 + T_4 + \ldots + T_{2n} = 4[T_1 + T_3 + \ldots + T_{2n-1}]$$

Let the G.P be $a, ar, ar^2, \ldots$

$$\frac{ar(r^n - 1)}{r-1} = \frac{4 \times a(r^n - 1)}{r-1}$$
$$\Rightarrow ar = 4a$$
$$\Rightarrow r = 4$$

Thus, the common ratio of the G.P is 4.

Question 12:
The sum of first four terms of an A.P is 56. The sum of the last four terms is 112. If its first term is 11, then find the number of terms.

Solution:

Let the A.P be $a, (a+d), (a+2d), (a+3d), \ldots a+(n-2)d, a+(n-1)d$
It is given that, $a = 11$

Sum of the first four terms

$$\Rightarrow a + (a+d) + (a+2d) + (a+3d) = 56$$
$$\Rightarrow 4a + 6d = 56$$
$$\Rightarrow 4 \times 11 + 6d = 56$$
$$\Rightarrow 6d = 56 - 44$$
$$\Rightarrow d = \frac{12}{6}$$
$$\Rightarrow d = 2$$

Sum of last four terms

$$\Rightarrow [a+(n-4)d]+[a+(n-3)d]+[a+(n-2)d]+[a+(n-1)d]=112$$
$$\Rightarrow 4a+(4n-10)d=112$$
$$\Rightarrow 4\times 11+(4n-10)\times 2=112$$
$$\Rightarrow 44+8n-20=112$$
$$\Rightarrow 8n=112-24$$
$$\Rightarrow n=\frac{88}{8}$$
$$\Rightarrow n=11$$

Thus, the number of terms of the A.P is 11.

Question 13:

If $\frac{a+bx}{a-bx}=\frac{b+cx}{b-cx}=\frac{c+dx}{c-dx}(x\neq 0)$, then show that a,b,c and d are in G.P.

Solution:

It is given that $\frac{a+bx}{a-bx}=\frac{b+cx}{b-cx}=\frac{c+dx}{c-dx}(x\neq 0)$,
Therefore,

$$\Rightarrow \frac{a+bx}{a-bx}=\frac{b+cx}{b-cx}$$
$$\Rightarrow (a+bx)(b-cx)=(b+cx)(a-bx)$$
$$\Rightarrow ab-acx+b^2x-bcx^2=ab-b^2x+acx-bcx^2$$
$$\Rightarrow 2b^2x=2acx$$
$$\Rightarrow b^2=ac$$
$$\Rightarrow \frac{b}{a}=\frac{c}{b} \qquad \ldots(1)$$

Also,

$$\Rightarrow \frac{b+cx}{b-cx}=\frac{c+dx}{c-dx}$$
$$\Rightarrow (b+cx)(c-dx)=(b-cx)(c+dx)$$
$$\Rightarrow bc-bdx+c^2x-cdx^2=bc+bdx-c^2x-cdx^2$$
$$\Rightarrow 2c^2x=2bdx$$
$$\Rightarrow c^2=bd$$
$$\Rightarrow \frac{c}{b}=\frac{d}{c} \qquad \ldots(2)$$

From (1) and (2), we obtain

$$\frac{b}{a}=\frac{c}{b}=\frac{d}{c}$$

Thus, a,b,c and d are in G.P.

Question 14:

Let S be the sum, P the product and R the sum of reciprocals of n terms in a G.P. Prove that $P^2R^n=S^n$.

Solution:

Let the G.P be $a,ar,ar^2,ar^3\ldots ar^{n-1}$

According to the given information

Sum of the n terms

$$S=\frac{a\left(r^n-1\right)}{r-1}$$

Product of the n terms

$$P = a^n \times r^{1+2+3+\ldots+n-1}$$

$$= a^n r^{\frac{n(n-1)}{2}} \qquad \left[\because \text{ sum of first } n \text{ natural numbers } = n\frac{(n+1)}{2}\right]$$

Sum of reciprocals of n terms

$$R = \frac{1}{a} + \frac{1}{ar} + \ldots + \frac{1}{ar^{n-1}}$$

$$= \frac{r^{n-1} + r^{n-2} + \ldots r + 1}{ar^{n-1}}$$

$$= \frac{1(r^n - 1)}{(r-1)} \times \frac{1}{ar^{n-1}} \qquad \left[\because 1, r, \ldots r^{n-1} \text{ forms a G.P}\right]$$

$$= \frac{r^n - 1}{ar^{n-1}(r-1)}$$

Therefore,

$$P^2 R^n = a^{2n} r^{n(n-1)} \frac{(r^n - 1)^n}{a^n r^{n(n-1)} (r-1)^n}$$

$$= \frac{a^n (r^n - 1)^n}{(r-1)^n}$$

$$= \left[\frac{a(r^n - 1)}{(r-1)}\right]^n$$

$$= S^n$$

Hence, $P^2 R^n = S^n$ proved

Question 15:

The p^{th}, q^{th} and r^{th} terms of an A.P are a, b, c respectively. Show that $(q-r)a + (r-p)b + (p-q)c = 0$.

Solution:

Let t and d be the first term and common difference of the A.P respectively.

The n^{th} term of an A.P is given by, $a_n = t + (n-1)d$

$$a_p = t+(p-1)d = a \qquad \ldots(1)$$
$$a_q = t+(q-1)d = b \qquad \ldots(2)$$
$$a_r = t+(r-1)d = c \qquad \ldots(3)$$

Subtracting (2) from (1), we obtain

$$\Rightarrow (p-1-q+1)d = a-b$$
$$\Rightarrow (p-q)d = a-b$$
$$\Rightarrow d = \frac{a-b}{p-q} \qquad \ldots(4)$$

Subtracting (3) from (2), we obtain

$$\Rightarrow (q-1-r+1)d = b-c$$
$$\Rightarrow (q-r)d = b-c$$
$$\Rightarrow d = \frac{b-c}{q-r} \qquad \ldots(5)$$

From (4) and (5), we obtain

$$\Rightarrow \frac{a-b}{p-q} = \frac{b-c}{q-r}$$
$$\Rightarrow (a-b)(q-r) = (b-c)(p-q)$$
$$\Rightarrow aq-ar-bq+br = bp-bq-cp+cq$$
$$\Rightarrow bp-cp+cq-aq+ar-br = 0$$
$$\Rightarrow (-aq+ar)+(bp-br)+(-cp+cq) = 0$$
$$\Rightarrow -a(q-r)-b(r-p)-c(p-q) = 0$$
$$\Rightarrow a(q-r)+b(r-p)+c(p-q) = 0$$

Hence, $(q-r)a+(r-p)b+(p-q)c = 0$ proved.

Question 16:

If $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in A.P, prove that a, b, c are in A.P.

Solution:

It is given that $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in A.P.

Therefore,

$$\Rightarrow b\left(\frac{1}{c}+\frac{1}{a}\right)-a\left(\frac{1}{b}+\frac{1}{c}\right)=c\left(\frac{1}{a}+\frac{1}{b}\right)-b\left(\frac{1}{c}+\frac{1}{a}\right)$$
$$\Rightarrow \frac{b(a+c)}{ac}-\frac{a(b+c)}{bc}=\frac{c(a+b)}{ab}-\frac{b(a+c)}{ac}$$
$$\Rightarrow \frac{b^2a+b^2c-a^2b-a^2c}{abc}=\frac{c^2a+c^2b-b^2a-b^2c}{abc}$$
$$\Rightarrow b^2a-a^2b+b^2c-a^2c=c^2a-b^2a+c^2b-b^2c$$
$$\Rightarrow ab(b-a)+c(b^2-a^2)=a(c^2-b^2)+bc(c-b)$$
$$\Rightarrow ab(b-a)+c(b+a)(b-a)=a(c-b)(c+b)+bc(c-b)$$
$$\Rightarrow (b-a)(ab+cb+ca)=(c-b)(ac+ab+bc)$$
$$\Rightarrow b-a=c-b$$

Thus, a,b,c are in A.P. proved.

Question 17:

If a,b,c,d are in G.P, prove that $(a^n+b^n),(b^n+c^n),(c^n+d^n)$ are in G.P.

Solution:

It is given that a,b,c,d are in G.P.
Therefore,

$$b^2=ac \qquad \ldots(1)$$
$$c^2=bd \qquad \ldots(2)$$
$$ad=bc \qquad \ldots(3)$$

We need to prove $(a^n+b^n),(b^n+c^n),(c^n+d^n)$ are in G.P. i.e.,

$$(b^n+c^n)^2=(a^n+b^n)\cdot(c^n+d^n)$$

Consider

$$\begin{aligned}(b^n+c^n)^2&=b^{2n}+2b^nc^n+c^{2n}\\ &=(b^2)^n+2b^nc^n+(c^2)^n\\ &=(ac)^n+2b^nc^n+(bd)^n && [\text{Using (1) and (2)}]\\ &=a^nc^n+b^nc^n+b^nc^n+b^nd^n\\ &=a^nc^n+b^nc^n+a^nd^n+b^nd^n && [\text{Using (3)}]\\ &=c^n(a^n+b^n)+d^n(a^n+b^n)\\ &=(a^n+b^n)(c^n+d^n)\end{aligned}$$

Hence, $\left(b^n+c^n\right)^2=\left(a^n+b^n\right)\cdot\left(c^n+d^n\right)$

Thus, $\left(a^n+b^n\right),\left(b^n+c^n\right),\left(c^n+d^n\right)$ are in G.P.

Question 18:

If a and b are the roots of $x^2-3x+p=0$ and c, d are the roots of $x^2-12x+q=0$ where a,b,c,d form a G.P. Prove that $(q+p):(q-p)=17:15$.

Solution:

It is given that a and b are the roots of $x^2-3x+p=0$

Therefore,

$$a+b=3, \quad ab=p \quad \ldots(1)$$

Also, c and d are the roots of $x^2-12x+q=0$

$$c+d=12, \quad cd=q \quad \ldots(2)$$

It is given that a,b,c,d form a G.P.

Let $a=x, b=xr, c=xr^2, d=xr^3$

Form (1) and (2), we obtain

$$\Rightarrow x+xr=3$$
$$\Rightarrow x(1+r)=3 \quad \ldots(3)$$

$$\Rightarrow xr^2+xr^3=12$$
$$\Rightarrow xr^2(1+r)=12 \quad \ldots(4)$$

On dividing (4) by (3), we obtain

$$\frac{xr^2(1+r)}{x(1+r)}=\frac{12}{3}$$
$$\Rightarrow r^2=4$$
$$\Rightarrow r=\pm 2$$

<u>Case I</u>: when $r=2, x=1$

$$ab = x^2r = 2,$$
$$cd = x^2r^5 = 32$$

Therefore,

$$\Rightarrow \frac{q+p}{q-p} = \frac{32+2}{32-2} = \frac{34}{30} = \frac{17}{15}$$
$$\Rightarrow (q+p):(q-p) = 17:15$$

Case II: when $r = -2, x = -3$

$$ab = x^2r = -18$$
$$cd = x^2r^5 = -288$$

Therefore,

$$\Rightarrow \frac{q+p}{q-p} = \frac{-288-18}{-288+18} = \frac{-306}{-270} = \frac{17}{15}$$
$$\Rightarrow (q+p):(q-p) = 17:15$$

Thus, $(q+p):(q-p) = 17:15$ proved

Question 19:

The ratio of the A.M and G.M of two positive numbers a and b, is $m:n$. Show that $a:b = \left(m+\sqrt{m^2-n^2}\right):\left(m-\sqrt{m^2-n^2}\right)$.

Solution:

Let the two numbers be a and b.

$$A.M = \frac{a+b}{2}$$

$$G.M = \sqrt{ab}$$

According to the given condition,

$$\Rightarrow \frac{a+b}{2\sqrt{ab}} = \frac{m}{n}$$
$$\Rightarrow \frac{(a+b)^2}{4ab} = \frac{m^2}{n^2}$$
$$\Rightarrow (a+b)^2 = \frac{4abm^2}{n^2}$$
$$\Rightarrow (a+b) = \frac{2m\sqrt{ab}}{n} \qquad \ldots(1)$$

Using this in the identity $(a-b)^2 = (a+b)^2 - 4ab$, we obtain

$$(a-b)^2=\frac{4abm^2}{n^2}-4ab$$
$$=\frac{4ab\left(m^2-n^2\right)}{n^2}$$
$$(a-b)=\frac{2\sqrt{ab}\sqrt{m^2-n^2}}{n} \qquad \ldots(2)$$

Adding (1) and (2), we obtain

$$2a=\frac{2\sqrt{ab}}{n}\left(m+\sqrt{m^2-n^2}\right)$$
$$a=\frac{\sqrt{ab}}{n}\left(m+\sqrt{m^2-n^2}\right)$$

Substituting the value of a in (1), we obtain

$$b=\frac{2\sqrt{ab}}{n}m-\frac{\sqrt{ab}}{n}\left(m+\sqrt{m^2-n^2}\right)$$
$$=\frac{\sqrt{ab}}{n}m-\frac{\sqrt{ab}}{n}\left(\sqrt{m^2-n^2}\right)$$
$$=\frac{\sqrt{ab}}{n}\left(m-\sqrt{m^2-n^2}\right)$$

Therefore,

$$\frac{a}{b}=\frac{\left(\frac{\sqrt{ab}}{n}\left(m+\sqrt{m^2-n^2}\right)\right)}{\left(\frac{\sqrt{ab}}{n}\left(m-\sqrt{m^2-n^2}\right)\right)}$$
$$=\frac{\left(m+\sqrt{m^2-n^2}\right)}{\left(m-\sqrt{m^2-n^2}\right)}$$

Thus, $a:b=\left(m+\sqrt{m^2-n^2}\right):\left(m-\sqrt{m^2-n^2}\right)$.

Question 20:

If a,b,c are in A.P; b,c,d are in G.P and $\frac{1}{c},\frac{1}{d},\frac{1}{e}$ are in A.P. Prove that a,c,e are in G.P.

Solution:

It is given that a, b, c are in A.P

$$b-a=c-b$$

$$b=\frac{a+c}{2} \quad \ldots(1)$$

It is given that b, c, d are in G.P

$$c^2=bd$$

$$c^2=\left(\frac{a+c}{2}\right)d \quad [\text{from } (1)]$$

$$d=\frac{2c^2}{a+c} \quad \ldots(2)$$

Also, $\frac{1}{c}, \frac{1}{d}, \frac{1}{e}$ are in A.P.

Therefore,

$$\Rightarrow \frac{1}{d}-\frac{1}{c}=\frac{1}{e}-\frac{1}{d}$$

$$\Rightarrow \frac{2}{d}=\frac{1}{c}+\frac{1}{e}$$

$$\Rightarrow \frac{2}{\frac{2c^2}{a+c}}=\frac{1}{c}+\frac{1}{e} \quad [\text{from } (2)]$$

$$\Rightarrow \frac{2(a+c)}{2c^2}=\frac{1}{c}+\frac{1}{e}$$

$$\Rightarrow \frac{(a+c)}{c^2}=\frac{e+c}{ce}$$

$$\Rightarrow \frac{(a+c)}{c}=\frac{e+c}{e}$$

$$\Rightarrow (a+c)e=(e+c)c$$

$$\Rightarrow ae+ce=ec+c^2$$

$$\Rightarrow c^2=ae$$

Thus, a, c, e are in G.P.

Question 21:

Find the sum of the following series up to n terms:

(i) $5+55+555+\ldots$

(ii) $.6+.66+.666+\ldots$

Solution:

(i) $5+55+555+\ldots$

Let $S_n=5+55+555+\ldots n$ terms

$$S_n = \frac{5}{9}[9 + 99 + 999 + \ldots n\text{terms}]$$

$$= \frac{5}{9}\left[(10 - 1) + \left(10^2 - 1\right) + \left(10^3 - 1\right) + \ldots n\text{terms}\right]$$

$$= \frac{5}{9}\left[\left(10 + 10^2 + 10^3 + \ldots n\text{terms}\right) - (1 + 1 + \ldots n\text{terms})\right]$$

$$= \frac{5}{9}\left[\frac{10\left(10^n - 1\right)}{10 - 1} - n\right]$$

$$= \frac{5}{9}\left[\frac{10\left(10^n - 1\right)}{9} - n\right]$$

$$= \frac{50}{81}\left(10^n - 1\right) - \frac{5n}{9}$$

(ii) $.6+.66+.666+\ldots$

Let $S_n = .6 + .66 + .666 + \ldots n\text{terms}$ $S_n = .6 + .66 + .666 + \ldots n\text{terms}$

$$S_n = 6[0.1 + 0.11 + 0.111 + \ldots n\text{terms}]$$
$$= \frac{6}{9}[0.9 + 0.99 + 0.999 + \ldots n\text{terms}]$$
$$= \frac{6}{9}\left[\left(1-\frac{1}{10}\right)+\left(1-\frac{1}{10^2}\right)+\left(1-\frac{1}{10^3}\right)+\ldots n\text{terms}\right]$$
$$= \frac{2}{3}\left[(1+1+1+\ldots n\text{terms}) - \frac{1}{10}\left(1+\frac{1}{10}+\frac{1}{10^2}+\ldots n\text{terms}\right)\right]$$
$$= \frac{2}{3}\left[n - \frac{1}{10}\left(\frac{1-\left(\frac{1}{10}\right)^n}{1-\frac{1}{10}}\right)\right]$$
$$= \frac{2}{3}n - \frac{2}{30}\times\frac{10}{9}\left(1-10^{-n}\right)$$
$$= \frac{2}{3}n - \frac{2}{27}\left(1-10^{-n}\right)$$

Question 22:

Find the 20^{th} term of the series .

Solution:

The given series is

Therefore,

$$a_n = 2n \times (2n+2)$$
$$= 4n^2 + 4n$$

$$a_{20} = 4(20)^2 + 4(20)$$
$$= 4(400) + 80$$
$$= 1680$$

Thus, the 20^{th} term of the series is 1680.

Question 23:

Find the sum of the first nn terms of the series: 3+7+13+21+31+...

Solution:

The given series is $3+7+13+21+31+\ldots$, can be written as

$$S=3+7+13+21+31+\ldots+a_{n-1}+a_n \qquad \ldots(1)$$
$$S=3+7+13+21+\ldots+a_{n-2}+a_{n-1}+a_n \qquad \ldots(2)$$

On subtracting the equation (2) from (1), we obtain

$$S-S=[3+7+13+21+31+\ldots+a_{n-1}+a_n]-[3+7+13+21+\ldots+a_{n-2}+a_{n-1}+a_n]$$
$$0=3+(7+13+21+31+\ldots+a_{n-1}+a_n)-(3+7+13+21+\ldots+a_{n-2}+a_{n-1})-a_n$$
$$0=3+[(7-3)+(13-7)+(21-13)+\ldots+(a_n-a_{n-1})]-a_n$$
$$0=3+[4+6+8+\ldots(n-1)\text{ terms}]-a_n$$
$$a_n=3+[4+6+8+\ldots(n-1)\text{ terms}]$$
$$=3+\left(\frac{n-1}{2}\right)[2\times4+(n-1-1)2]$$
$$=3+\left(\frac{n-1}{2}\right)[8+2n-4]$$
$$=3+\left(\frac{n-1}{2}\right)[2n+4]$$
$$=3+(n-1)(n+2)$$
$$=3+(n^2+n-2)$$
$$a_n=n^2+n+1$$

Therefore

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} k^2 + k + 1$$

$$= \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + n$$

$$= n\left[\frac{(n+1)(2n+1) + 3(n+1) + 6}{6}\right]$$

$$\sum_{k=1}^{n} a_k = n\left[\frac{2n^2 + 3n + 1 + 3n + 3 + 6}{6}\right]$$

$$= n\left[\frac{2n^2 + 6n + 10}{6}\right]$$

$$= \frac{n}{3}(n^2 + 3n + 5)$$

Question 24:

If S_1, S_2, S_3 are the sums of first n natural numbers, their squares and their cubes respectively, show that $9S_2^2 = S_3(1 + 8S_1)$.

Solution:

From the given information,

$$S_1 = \frac{n(n+1)}{2}$$

$$S_2 = \frac{n(n+1)(2n+1)}{6}$$

$$S_3 = \left[\frac{n(n+1)}{2}\right]^2$$

Hence,

$$9{S_2}^2 = 9\left[\frac{n(n+1)(2n+1)}{6}\right]^2$$
$$= \frac{9}{36}[n(n+1)(2n+1)]^2$$
$$= \frac{1}{4}[n(n+1)(2n+1)]^2$$
$$= \left[\frac{n(n+1)(2n+1)}{2}\right]^2 \qquad \ldots(1)$$

Also,

$$S_3(1+8S_1) = \left[\frac{n(n+1)}{2}\right]^2\left[1+\frac{8n(n+1)}{2}\right]$$
$$= \left[\frac{n(n+1)}{2}\right]^2[1+4n^2+4n]$$
$$= \left[\frac{n(n+1)}{2}\right]^2(2n+1)^2$$
$$= \left[\frac{n(n+1)(2n+1)}{2}\right]^2 \qquad \ldots(2)$$

Thus, from (1) and (2), we obtain

$$9S_2^2 = S_3(1+8S_1)$$

Question 25:

Find the sum of the following series up to n terms:

$$\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\ldots$$

Solution:

The n^{th}term of the given series $\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\ldots$ is

$$\begin{aligned} a_n &= \frac{1^3+2^3+3^3+\ldots n^3}{1+3+5+\ldots+(2n-1)} \\ &= \frac{\left[\frac{n(n+1)}{2}\right]^2}{1+3+5+\ldots+(2n-1)} \end{aligned}$$

Here, $1,3,5,\ldots(2n-1)$ is an A.P. with first term a, last term $(2n-1)$ and number of terms n.

Therefore,

$$\begin{aligned} 1+3+5+\cdots+(2n-1) &= \frac{n}{2}[2\times 1+(n-1)2] \\ &= n^2 \end{aligned}$$

Hence,

$$\begin{aligned} a_n &= \frac{n^2(n+1)^2}{4n^2} \\ &= \frac{n^2+2n+1}{4} \\ &= \frac{1}{4}n^2+\frac{1}{2}n+\frac{1}{4} \end{aligned}$$

Thus,

$$\begin{aligned} S_n &= \sum_{k=1}^{n} a_k \\ Z &= \sum_{k=1}^{n}\left(\frac{1}{4}k^2+\frac{1}{2}k+\frac{1}{4}\right) \\ &= \frac{1}{4}\left[\frac{n(n+1)(2n+1)}{6}\right]+\frac{1}{2}\left[\frac{n(n+1)}{2}\right]+\frac{1}{4}n \\ &= \frac{n[(n+1)(2n+1)+6(n+1)+6]}{24} \\ &= \frac{n[2n^2+3n+1+6n+6+6]}{24} \\ &= \frac{n}{24}(2n^2+9n+13) \end{aligned}$$

Question 26:

Show that $\frac{1\times2^2+2\times3^2+\ldots+n\times(n+1)^2}{1^2\times2+2^2\times3+\ldots+n^2\times(n+1)}=\frac{3n+5}{3n+1}$.

Solution:

The n^{th} term of the given series $1\times2^2+2\times3^2+\ldots+n\times(n+1)^2$ in numerator

$$\begin{aligned}a_n&=n(n+1)^2\\&=n^3+2n^2+n\end{aligned}$$

The n^{th} term of the given series $1^2\times2+2^2\times3+\ldots+n^2\times(n+1)$ in denominator

$$\begin{aligned}a_n&=n^2(n+1)\\&=n^3+n^2\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1\times2^2+2\times3^2+\ldots+n\times(n+1)^2}{1^2\times2+2^2\times3+\ldots+n^2\times(n+1)}&=\frac{\sum_{k=1}^{n}a_k}{\sum_{k=1}^{n}a_k}\\&=\frac{\sum_{k=1}^{n}(k^3+2k^2+k)}{\sum_{k=1}^{n}(k^3+k^2)}\qquad\ldots(1)\end{aligned}$$

Here,

$$\begin{aligned}\sum_{k=1}^{n}(k^3+2k^2+k)&=\frac{n^2(n+1)^2}{4}+\frac{2n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}\\&=\frac{n(n+1)}{2}\left[\frac{n(n+1)}{2}+\frac{2}{3}(2n+1)+1\right]\\&=\frac{n(n+1)}{2}\left[\frac{3n^2+3n+8n+4+6}{6}\right]\\&=\frac{n(n+1)}{12}[3n^2+11n+10]\\&=\frac{n(n+1)}{12}[3n^2+6n+5n+10]\\&=\frac{n(n+1)}{12}[3n^2(n+2)+5(n+2)]\\&=\frac{n(n+1)(n+2)(3n+5)}{12}\qquad\ldots(2)\end{aligned}$$

Also,

$$\sum_{k=1}^{n}\left(k^3+k^2\right)=\frac{n^2(n+1)^2}{4}+\frac{n(n+1)(2n+1)}{6}$$

$$=\frac{n(n+1)}{2}\left[\frac{n(n+1)}{2}+\frac{2n+1}{3}\right]$$

$$=\frac{n(n+1)}{2}\left[\frac{3n^2+3n+4n+2}{6}\right]$$

$$=\frac{n(n+1)}{12}\left[3n^2+7n+2\right]$$

$$=\frac{n(n+1)}{12}\left[3n^2+6n+n+2\right]$$

$$=\frac{n(n+1)}{12}\left[3n(n+2)+1(n+2)\right]$$

$$=\frac{n(n+1)(n+2)(3n+1)}{12} \qquad \ldots(3)$$

From (1), $(2)$and (3), we obtain

$$\frac{1\times 2^2+2\times 3^2+\ldots+n\times(n+1)^2}{1^2\times 2+2^2\times 3+\ldots+n^2\times(n+1)}=\frac{\frac{n(n+1)(n+2)(3n+5)}{12}}{\frac{n(n+1)(n+2)(3n+1)}{12}}$$

$$=\frac{n(n+1)(n+2)(3n+5)}{n(n+1)(n+2)(3n+1)}$$

$$=\frac{3n+5}{3n+1}$$

Thus, $\frac{1\times 2^2+2\times 3^2+\ldots+n\times(n+1)^2}{1^2\times 2+2^2\times 3+\ldots+n^2\times(n+1)}=\frac{3n+5}{3n+1}$ proved.

Question 27:

A farmer buys a used tractor for ₹ 12000. He pays ₹ 6000 cash and agrees to pay the balance in annual instalment of ₹ 500 plus 12%interest on the unpaid amount. How much will the tractor cost him?

Solution:

It is given that farmer pays ₹ 6000 in cash.

Hence, unpaid amount in ₹ $= 12000 - 6000 = 6000$

According to the given condition, the interest paid annually is

$$(12\% of 6000), (12\% of 5500), (12\% of 5000), \ldots (12\% of 500)$$

Thus, total interest to be paid $500 + 1000 + \ldots + 6000$

$$\begin{aligned} &= 12\% of 6000 + 12\% of 5500 + \ldots + 12\% of 500 \\ &= 12\% of (6000 + 5500 + \ldots + 500) \\ &= 12\% of (500 + 1000 + \ldots + 6000) \end{aligned}$$

Now, the series $500 + 1000 + \ldots + 6000$ forms an A.P. with a and d both equal to 500.

Let the number of terms of the A.P. be n

Therefore,

$$\begin{aligned} &\Rightarrow a_n = a + (n-1)d \\ &\Rightarrow 6000 = 500 + (n-1)(500) \\ &\Rightarrow 6000 - 500 = 500(n-1) \\ &\Rightarrow n - 1 = \frac{5500}{500} \\ &\Rightarrow n = 11 + 1 \\ &\Rightarrow n = 12 \end{aligned}$$

Hence,

$$\begin{aligned} S_n &= \frac{n}{2}[2a + (n-1)d] \\ S_{12} &= \frac{12}{2}[2(500) + (12-1)(500)] \\ &= 6[1000 + 5500] \\ &= 6 \times 6500 \\ &= 39000 \end{aligned}$$

Therefore, total interest to be paid is

$$\begin{aligned} 12\% of\,(500 + 1000 + \ldots + 6000) &= 12\% of 39000 \\ &= \frac{12}{100} \times 39000 \\ &= 4680 \end{aligned}$$

Now, total cost of tractor in ₹ $= 12000 + 4680 = 16680$

Thus, the tractor will cost him ₹ 16680.

Question 28:

Shamshad Ali buys a scooter for ₹22000. He pays ₹4000cash and agrees to pay the balance in annual instalment of ₹1000 plus 10% interest on the unpaid amount. How much will the scooter cost him?

Solution:

It is given that Shamshad Ali buys a scooter for ₹22000 and pays ₹4000 in cash.

Hence, unpaid amount in ₹ $= 22000 - 4000 = 18000$

According to the given condition, the interest paid annually is

$(10\% of 18000)$, $(10\% of 17000)$, $(10\% of 16000)$, . . . $(10\% of 1000)$

Thus, total interest to be paid

$$\begin{aligned} &= (10\%\text{of}18000) + (10\%\text{of}17000) + \ldots + (10\% of 1000) \\ &= 10\% of\,(18000 + 17000 + \ldots + 1000) \\ &= 10\% of\,(1000 + 2000 + \ldots + 18000) \end{aligned}$$

Now, the series $1000 + 2000 + \ldots + 18000$forms an A.P. with a and d both equal to 1000.

Let the number of terms of the A.P. be n

Therefore,

$$\Rightarrow a_n = a + (n-1)\,d$$
$$\Rightarrow 18000 = 1000 + (n-1)\,(1000)$$
$$\Rightarrow 18000 - 1000 = 1000\,(n-1)$$
$$\Rightarrow n - 1 = \frac{17000}{1000}$$
$$\Rightarrow n = 17 + 1$$
$$\Rightarrow n = 18$$

Hence,

$$S_n = \frac{n}{2}[2a + (n-1)\,d]$$
$$S_{18} = \frac{18}{2}[2\,(1000) + (18-1)\,(1000)]$$
$$= 9\,[2000 + 17000]$$
$$= 9 \times 19000$$
$$= 171000$$

Therefore, total interest to be paid is

$$10\% \text{ of } (1000 + 2000 + \ldots + 18000) = 10\% of 171000$$
$$= \frac{10}{100} \times 171000$$
$$= 17100$$

Now, total cost of scooter in ₹ $= 22000 + 17100 = 39100$

Thus, the scooter will cost him ₹39100.

Question 29:

A person writes a letter to four of his friends. He asks each one of them to copy the letter and mail to four different persons with instruction that they move the chain similarly. Assuming that the chain is not broken and that it costs 50 paise to mail one letter, find the amount spent on the postage when 8^{th} set of letter is mailed.

Solution:

The number of letters mailed forms a G.P: $4, 4^2, \ldots, 4^8$

Here, $a = 4, r = 4, n = 8$

Hence,

$$S_n = \frac{a(r^n - 1)}{r - 1}$$
$$S_8 = \frac{4(4^8 - 1)}{4 - 1}$$
$$= \frac{4(65536 - 1)}{3}$$
$$= \frac{4 \times 65535}{3}$$
$$= 4 \times 21845$$
$$= 87380$$

It is given that the cost to mail one letter is 50 paisa

Therefore, cost in ₹ for mailing 87380 letters $= \frac{50}{100} \times 87380 = 43690$

Thus, ₹43690spent when on the postage when 8^{th} set of letter is mailed.

Question 30:

A man deposited ₹10000 in a bank at the rate of 5% simple interest annually. Find the amount in the 15th year since he deposited the amount and also calculate the total amount after 20 years.

Solution:

It is given that the man deposited ₹10000in a bank at the rate of 5% simple interest annually

Hence, annual interest received in ₹ $= \frac{5}{100} \times 10000 = 500$

Amount in ₹ at the end of first year $= 10000 + 500 = 10500$

Amount in ₹ at the end of second year $= 10500 + 500 = 11000$

Amount in ₹ at the end of third year $= 11000 + 500 = 11500$

Therefore, $10000, 10500, 11000, \ldots$ is an A.P.

Hence, the amount in the 15^{th} year

$$\begin{aligned} a_{15} &= 10000 + (15-1)(500) \\ &= 10000 + 7000 \\ &= 17000 \end{aligned}$$

Amount after 20 years

$$\begin{aligned} a_{21} &= 10000 + (21-1)(500) \\ &= 10000 + 10000 \\ &= 20000 \end{aligned}$$

Thus, the amount in 15^{th} year is ₹17000 and after 20 years amount is ₹20000.

Question 31:

A manufacturer reckons that the value of a machine, which costs him ₹15625, will depreciate each year by 20%. Find the estimated value at the end of 5 years.

Solution:

Cost of machine is ₹15625

Machine depreciates by 20% every year.

Therefore,

its value after every year is 80 of the original cost i.e., $\frac{4}{5}$of the original cost.

Value at end of 5 years $= 15625 \times \underbrace{\frac{4}{5} \times \frac{4}{5} \times \frac{4}{5} \times \ldots \times \frac{4}{5}}_{5 times} = 15625 \times \left(\frac{4}{5}\right)^5 = 5 \times 1024 = 5120$

Thus, value of the machine at the end of 5 years is ₹ 5120.

Question 32:

150 workers were engaged to finish a job in a certain number of days. 4 workers dropped out on second day, 4 more workers dropped out on third day and so on. It took 8 more days to finish the work. Find the number of days in which the work was completed.

Solution:

Let x be the number of days in which 150 workers finish the work.

According to the given information,

$$150x = 150 + 146 + 142 + \ldots (x+8) \text{ terms}$$

Here, $150 + 146 + 142 + \ldots (x+8)$ terms forms an A.P. with $a = 146, d = -4, n = (x+8)$

Therefore

$$\Rightarrow 150x = \frac{(x+8)}{2}[2(150) + (x+8-1)(-4)]$$
$$\Rightarrow 150x = (x+8)[150 + (x+7)(-2)]$$
$$\Rightarrow 150x = (x+8)(150 - 2x - 14)$$
$$\Rightarrow 150x = (x+8)(136 - 2x)$$
$$\Rightarrow 75x = (x+8)(68 - x)$$
$$\Rightarrow 75x = 68x - x^2 + 544 - 8x$$
$$\Rightarrow x^2 + 15x - 544 = 0$$
$$\Rightarrow x^2 + 32x - 17x - 544 = 0$$
$$\Rightarrow x(x+32) - 17(x+32) = 0$$
$$\Rightarrow (x-17)(x+32) = 0$$
$$\Rightarrow x = 17, -32$$

However, x cannot be negative, hence, $x = 17$

Therefore, the number of days in which the work was completed is $(17+8) = 25$ days.

Thus, the work was completed in 25 days.