Miscellaneous Exercise

Q1. Find a, b and n in the expansion of $(a + b)^n$ if the first three terms of the expansion are 729, 7290 and 30375, respectively.

A.1. The general term of the expansion $(a + b)^n$ is given by

$T_{r+1} = {}^nC_r a^{n-r} b^r$

So, $T_1 = {}^nC_0 a^n = a^n$

$$T_2 = {}^nC_1 a^{n-1} b = \frac{n!}{1!(n-1)!} a^{n-1} b = \frac{n \times (n-1)!}{(n-1)!} a^{n-1} b = n a^{n-1} b$$

$$T_3 = {}^nC_2 a^{n-2} b^2 = \frac{n!}{2!(n-2)!} a^{n-2} b^2 = \frac{n \times (n-1) \times (n-2)!}{2 \times 1 \times (n-2)!} a^{n-2} b^2 = \frac{n(n-1)}{2} a^{n-2} b^2$$

Given,

$T_1 = 729$

$\Rightarrow a^n = 729$ ------------------- (1)

$T_2 = 7290$

$\Rightarrow n a^{n-1} b = 7290$ ------------- (2)

$T_3 = 30375$

$\Rightarrow \frac{n(n-1)}{2} a^{n-2} b^2 = 30375$ -------------------- (3)

Dividing equation (2) by (1) we get,

$$\frac{n a^{n-1} b}{a^n} = \frac{7290}{729}$$

$$\Rightarrow \frac{nb}{a} = 10$$

Similarly dividing equation (3) by (2) we get,

$$\frac{n(n-1)}{2} a^{n-2}b^2 \div na^{n-1}b = \frac{30375}{7290}$$

$$\Rightarrow \frac{n(n-1)}{2} a^{n-2}b^2 \times \frac{1}{na^{n-1}b} = \frac{30375}{7290}$$

$$\Rightarrow n(n-1)\, a^{n-2}b^2 \times \frac{1}{na^{n-1}b} = \frac{30375}{7290} \times 2$$

$$\Rightarrow \frac{(n-1)\,b}{a} = \frac{25\times 2}{6}$$

$$\Rightarrow \frac{nb}{a} - \frac{b}{a} = \frac{25}{3}$$

$$\Rightarrow 10 - \frac{b}{a} = \frac{25}{3} \quad \left[\text{since, } \frac{nb}{a} = 10\right]$$

$$\Rightarrow \frac{b}{a} = 10 - \frac{25}{3}$$

$$= \frac{30-25}{3}$$

$$= \frac{5}{3} \quad \text{---------------- (5)}$$

Putting equation (5) in (4) we get,

$$n \times \frac{5}{3} = 10$$

$$\Rightarrow n = 10 \times \frac{3}{5}$$

$$\Rightarrow n = 6$$

So putting the value of n in equation (1) we get,

$$a^6 = 729$$

$$\Rightarrow a^6 = 3^6$$

$$\Rightarrow a = 3$$

And putting $a = 3$ in equation (5) we get,

$$\frac{b}{a} = \frac{5}{3}$$

$$\Rightarrow b = \frac{5}{3} \times a$$

$$= \frac{5}{3} \times 3$$

$$= 5$$

Q2. **Find a if the coefficients of x^2 and x^3 in the expansion of $(3 + ax)^9$ are equal.**

A.2. The general term of the expansion $(3 + ax)^9$ is

$$T_{r+1} = {}^9C_r\, 3^{9-r} \quad ax^{\;r}$$

$$= {}^9C_r\, 3^{9-r}\, a^r x^r$$

At $r = 2$,

$$T_{2+1} = {}^9C_2\, 3^{9-2}\, a^2 x^2$$

$$= \frac{9!}{2!\ 9-2\ !} 3^7 a^2 x^2$$

$$= \frac{9 \times 8 \times 7!}{2 \times 1 \times 7!} 3^7 a^2 x^2$$

$$= 36 \times 3^7 a^2 x^2$$

At r = 3,

$$T_{3+1} = {}^9C_3\, 3^{9-3}\, a^3 x^3$$

$$= \frac{9!}{3!\ 9-3\ !} 3^6 a^3 x^3$$

$$= \frac{9 \times 8 \times 7 \times 6!}{3 \times 2 \times 1 \times 6!} 3^6 a^3 x^3$$

$$= 84 \times 3^6 a^3 x^3.$$

Given that,

Co-efficient of x^2 = co-efficient of x^3

=> $36 \times 3^7 a^2 = 84 \times 3^6 a^3$

=> $\frac{a^3}{a^2} = \frac{36 \times 3^7}{84 \times 3^6}$

=> $a = \frac{3 \times 3}{7}$

$$= \frac{9}{7}$$

Q3. Find the coefficient of x^5 in the product $(1 + 2x)^6(1 – x)^7$ using binomial theorem.

A.3. We first expand each of the factors of the given product using Binomial theorem. We have,

$(1 + 2x)^6 = {}^6C_0\,(1)^6 + {}^6C_1\,(1)^5(2x) + {}^6C_2\,(1)^4(2x)^2 + {}^6C_3\,(1)^3(2x)^3 + {}^6C_4\,(1)^2(2x)^4 + {}^6C_5\,(1)(2x)^5 + {}^6C_6(2x)^6$

$$= \left[\frac{6!}{0!\ 6-0\ !} \times 1\right] + \left[\frac{6!}{1!\ 6-1\ !} \times 1 \times\ 2x\right] + \left[\frac{6!}{2!\ 6-2\ !} \times 1 \times 4x^2\right] + \left[\frac{6!}{3!\ 6-3\ !} \times 1 \times 8x^3\right] +$$

$$\left[\frac{6!}{4!\ 6-4\ !} \times 1 \times 16x^4\right] + \left[\frac{6!}{5!\ 6-5\ !} \times 1 \times 32x^5\right] + \left[\frac{6!}{6!\ 6-6\ !} \times 64x^6\right]$$

$= [1] + \left[\frac{6\times5!}{1\times5!}\times(2x)\right] + \left[\frac{6\times5\times4!}{2\times1\times4!}\times4x^2\right] + \left[\frac{6\times5\times4\times3!}{3\times2\times1\times3!}\times 8x^3\right] + \left[\frac{6\times5\times4!}{4!\times2!}\times16x^4\right] +$
$\left[\frac{6\times5!}{5!\times1!}\times(32x^5)\right] + [1\times64x^6]$

$= 1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6$

And,

$(1-x)^7 = {}^7C_0\,(1)^7 + {}^7C_1\,(1)^6(-x) + {}^7C_2\,(1)^5(-x)^2 + {}^7C_3\,(1)^4(-x)^3 + {}^7C_4\,(1)^3(-x)^4 + {}^7C_5\,(1)^2(-x)^5 + {}^7C_6\,(1)\,(-x)^6 + {}^7C_7(-x)^7$

$= \left[\frac{7!}{0!\,(7-0)!}\times1\right] + \left[\frac{7!}{1!\,(7-1)!}\times1\times(-x)\right] + \left[\frac{7!}{2!\,(7-2)!}\times1\times x^2\right] + \left[\frac{7!}{3!\,(7-3)!}\times1\times(-x)^3\right] +$
$\left[\frac{7!}{4!\,(7-4)!}\times1\times x^4\right] + \left[\frac{7!}{5!\,(7-5)!}\times1\times(-x)^5\right] + \left[\frac{7!}{6!\,(7-6)!}\times1\times x^6\right] + \left[\frac{7!}{7!\,(7-7)!}\times(-x)^7\right]$

$= [1] + \left[\frac{7\times6!}{1\times6!}\times(-x)\right] + \left[\frac{7\times6\times5!}{2\times1\times5!}\times x^2\right] + \left[\frac{7\times6\times5\times4!}{3\times2\times1\times4!}\times(-x^3)\right] + \left[\frac{7\times6\times5\times4!}{4!\times3!}\times x^4\right] +$
$\left[\frac{7\times6\times5!}{5!\times2!}\times(-x^5)\right] + \left[\frac{7\times6!}{6!\times1!}\times x^6\right] + [1\times(-x)^7]$

$= 1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7$

Thus, $(1+2x)^6(1-x)^7$

$= (1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6)(1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7)$

We need to find only the term involving x^5 i.e. $x^r x^{5-r}$

The terms with x^5 are

$= (1)(-21x^5) + (12x)(35x^4) + (60x^2)(-35x^3) + (160x^3)(21x^2) + (240x^4)(-7x) + (192x^5)(1)$

$= -21x^5 + 420x^5 - 2100x^5 + 3360x^5 - 1680x^5 + 192x^5$

$= 3972x^5 - 3801x^5$

$= 171x^5$

Hence, co-efficient of x^5 is 171.

Q4. If *a* and *b* are distinct integers, prove that *a* – *b* is a factor of $a^n - b^n$, whenever *n* is a positive integer.

[**Hint:** write $a^n = (a - b + b)^n$ and expand]

A.4. For (a – b) to be a factor of $a^n - b^n$ we need to show $(a^n - b^n) = (a - b)k$ as k is a natural number.

We have, for positive n

$a^n = (a-b+b)^n = [(a-b)+b]^n$

$\Rightarrow a^n = {}^nC_0(a-b)^n + {}^nC_1(a-b)^{n-1}b + {}^nC_2(a-b)^{n-2}b^2 + \ldots\ldots\ldots + {}^nC_{n-1}(a-b)\,b^{n-1} + {}^nC_n b^n$

$\Rightarrow a^n = (a-b)^n + {}^nC_1(a-b)^{n-1}b + {}^nC_2(a-b)^{n-2}b^2 + \ldots\ldots\ldots + {}^nC_{n-1}(a-b)\,b^{n-1} + b^n$

[Since, ${}^nC_0 = 1$ and ${}^nC_n = 1$]

$\Rightarrow a^n - b^n = (a-b)^n + {}^nC_1(a-b)^{n-1}b + {}^nC_2(a-b)^{n-2}b^2 + \ldots\ldots\ldots + {}^nC_{n-1}(a-b)\,b^{n-1}$

$\Rightarrow a^n - b^n = (a-b)\,[(a-b)^{n-1} + {}^nC_1 (a-b)^{n-2} b + {}^nC_2 (a-b)^{n-3} b^2 + \ldots\ldots\ldots\ldots + {}^nC_{n-1} b^{n-1}]$

$\Rightarrow a^n - b^n = (a-b)k$ where $k = [(a-b)^{n-1} + {}^nC_1 (a-b)^{n-2} b + {}^nC_2 (a-b)^{n-3} b^2 + \ldots\ldots\ldots\ldots + {}^nC_{n-1} b^{n-1}]$ is a natural number.

Therefore $(a - b)$ is a factor of $a^n - b^n$ where n is positive integer.

Q5. **Evaluate** $\left(\sqrt{3}+\sqrt{2}\right)^6 - \left(\sqrt{3}-\sqrt{2}\right)^6$.

A.5. Using binomial theorem on

$(a+b)^6 = {}^6C_0 (a)^6 + {}^6C_1 (a)^5(b) + {}^6C_2 (a)^4(b)^2 + {}^6C_3 (a)^3(b)^3 + {}^6C_4 (a)^2(b)^4 + {}^6C_5 (a)(b)^5 + {}^6C_6 (b)^6$

And

$(a-b)^6 = {}^6C_0 (a)^6 + {}^6C_1 (a)^5(-b) + {}^6C_2 (a)^4(-b)^2 + {}^6C_3 (a)^3(-b)^3 + {}^6C_4 (a)^2(-b)^4 + {}^6C_5 (a)(-b)^5 + {}^6C_6 (-b)^6$

$\Rightarrow (a-b)^6 = {}^6C_0 (a)^6 - {}^6C_1 (a)^5(b) + {}^6C_2 (a)^4(b)^2 - {}^6C_3 (a)^3(b)^3 + {}^6C_4 (a)^2(b)^4 - {}^6C_5 (a)(b)^5 + {}^6C_6 (b)^6$

Thus,

$(a+b)^6 - (a-b)^6$

$= [{}^6C_0 (a)^6 + {}^6C_1 (a)^5(b) + {}^6C_2 (a)^4(b)^2 + {}^6C_3 (a)^3(b)^3 + {}^6C_4 (a)^2(b)^4 + {}^6C_5 (a)(b)^5 + {}^6C_6 (b)^6] - [{}^6C_0 (a)^6 - {}^6C_1 (a)^5(b) + {}^6C_2 (a)^4(b)^2 - {}^6C_3 (a)^3(b)^3 + {}^6C_4 (a)^2(b)^4 - {}^6C_5 (a)(b)^5 + {}^6C_6 (b)^6]$

$= {}^6C_0 (a)^6 + {}^6C_1 (a)^5(b) + {}^6C_2 (a)^4(b)^2 + {}^6C_3 (a)^3(b)^3 + {}^6C_4 (a)^2(b)^4 + {}^6C_5 (a)(b)^5 + {}^6C_6 (b)^6 - {}^6C_0 (a)^6 + {}^6C_1 (a)^5(b) - {}^6C_2 (a)^4(b)^2 + {}^6C_3 (a)^3(b)^3 - {}^6C_4 (a)^2(b)^4 + {}^6C_5 (a)(b)^5 - {}^6C_6 (b)^6$

$= 2 \times [{}^6C_1 (a)^5(b) + {}^6C_3 (a)^3(b)^3 + {}^6C_5 (a)(b)^5]$

$= 2 \times \left[\frac{6!}{1!\,(6-1)!}(a)^5(b) + \frac{6!}{3!\,(6-3)!}(a)^3(b)^3 + \frac{6!}{5!\,(6-5)!}(a)(b)^5\right]$

$= 2 \times \left[\frac{6\times 5!}{1\times 5!}(a)^5(b) + \frac{6\times 5\times 4\times 3!}{3\times 2\times 1\times 3!}(a)^3(b)^3 \, \frac{6\times 5!}{5!\times 1!}(a)(b)^5\right]$

$= 2 \times [6(a)^5(b) + 20\,(a)^3(b)^3 + 6\,(a)(b)^5]$

Putting $a = \sqrt{3}$ and $b = \sqrt{2}$, we get

$(\sqrt{3}+\sqrt{2})^6 - (\sqrt{3}-\sqrt{2})^6$

$= 2\left[6(\sqrt{3})^5(\sqrt{2}) + 20(\sqrt{3})^3(\sqrt{2})^3 + 6(\sqrt{3})(\sqrt{2})^5\right]$

$= 2\left[6(9\sqrt{3})(\sqrt{2}) + 20(3\sqrt{3})(2\sqrt{2}) + 6(\sqrt{3})(4\sqrt{2})\right]$

$= 2\left[54\sqrt{6} + 120\sqrt{6} + 24\sqrt{6}\right]$

$= 2\left[198\sqrt{6}\right]$

$= 396\sqrt{6}$

Q6. Find the value of $\left(a^2+\sqrt{a^2-1}\right)^4 = \left(a^2-\sqrt{a^2-1}\right)^4$.

A.6. Using binomial theorem on

$$(x+y)^4 = {}^4C_0\, x^4 + {}^4C_1\, x^3y + {}^4C_2\, x^2y^2 + {}^4C_3\, xy^3 + {}^4C_4\, y^4$$

And $(x-y)^4 = {}^4C_0\, x^4 - {}^4C_1\, x^3y + {}^4C_2\, x^2y^2 - {}^4C_3\, xy^3 + {}^4C_4\, y^4$

Thus, $(x+y)^4 + (x-y)^4$

$$= {}^4C_0\, x^4 + {}^4C_1\, x^3y + {}^4C_2\, x^2y^2 + {}^4C_3\, xy^3 + {}^4C_4\, y^4 + {}^4C_0\, x^4 - {}^4C_1\, x^3y + {}^4C_2\, x^2y^2 - {}^4C_3\, xy^3 + {}^4C_4\, y^4$$

$$= 2\,[{}^4C_0\, x^4 + {}^4C_2\, x^2y^2 + {}^4C_4\, y^4]$$

$$= 2\left[\frac{4!}{0!\,(4-0)!}\,x^4 + \frac{4!}{2!\,(4-2)!}\,x^2y^2 + \frac{4!}{4!\,(4-4)!}\,y^4\right]$$

$$= 2\left[x^4 + \frac{4\times 3\times 2!}{2\times 2!}x^2y^2 + y^4\right]$$

$$= 2\,[x^4 + 6\,x^2y^2 + y^4]$$

Putting $x = a^2$ and $y = \sqrt{a^2-1}$, we get

$$(a^2+\sqrt{a^2-1})^4 + (a^2-\sqrt{a^2-1})^4$$

$$= 2\left[(a^2)^4 + 6\,(a^2)^2\left(\sqrt{a^2-1}\right)^2 + \left(\sqrt{a^2-1}\right)^4\right]$$

$$= 2\,[a^8 + 6a^4(a^2-1) + (a^2-1)^2]$$

$$= 2\,[a^8 + 6a^6 - 6a^4 + a^4 - 2a^2 + 1]$$

$$= 2\,[a^8 + 6a^6 - 5a^4 - 2a^2 + 1]$$

$$= 2a^8 + 12a^6 - 10a^4 - 4a^2 + 2$$

Q7. **Find an approximation of $(0.99)^5$ using the first three terms of its expansion.**

A.7. $(0.99)^5 = (1-0.01)^5$

By binomial theorem expanding upto first three terms, we get

$$(1-0.01)^5 = {}^5C_0\,(1)^5 + {}^5C_1\,(1)^4(-0.01) + {}^5C_2\,(1)^3(-0.01)^2$$

$$= \left(\frac{5!}{0!\,(5-0)!}\times 1\right) + \left[\frac{5!}{1!\,(5-1)!}\times 1\times(-0.01)\right] + \left(\frac{5!}{2!\,(5-2)!}\times 1\times 0.0001\right)$$

$$= (1\times 1) - \left(\frac{5\times 4!}{1\times 4!}\times 1\times 0.01\right) + \left(\frac{5\times 4\times 3!}{2\times 1\times 3!}\times 1\times 0.0001\right)$$

$$= 1 - 0.05 + 0.001$$

$$= 1.001 - 0.05$$

$$= 0.951$$

Q8. Find n, if the ratio of the fifth term from the beginning to the fifth term from theend in the expansion of $\left(\sqrt[4]{2}+\frac{1}{\sqrt[4]{3}}\right)^n$ is $\sqrt{6:1}$.

A.8. The 5th term of the expansion $\left(\sqrt[4]{2}+\frac{1}{\sqrt[4]{3}}\right)^n$ is given by

$$T_5 = T_{4+1} = {}^nC_4 \; \sqrt[4]{2}^{\;n-4}\left(\frac{1}{\sqrt[4]{3}}\right)^4$$

$$= \frac{n!}{4!\; n-4\;!}\;\frac{\sqrt[4]{2}^{\;n}}{\sqrt[4]{2}^{\;4}}\;\frac{1}{\sqrt[4]{3}^{\;4}}$$

$$= \frac{n!}{4!\; n-4\;!}\;\frac{\sqrt[4]{2}^{\;n}}{2\times 3}$$

$$= \frac{n!}{4!\; n-4\;!}\;\frac{\sqrt[4]{2}^{\;n}}{6}$$

We know that the r^{th} term from end of expansion $(x+a)^n$ will be the $(n-r+2)^{th}$ term and is given by

$${}^nC_{n-r+1}\; x^{r-1}\, a^{n-r+1}$$

So the last 5th term of the expansion $\left(\sqrt[4]{2}+\frac{1}{\sqrt[4]{3}}\right)^n$ is

$${}^nC_{n-5+1}\; \sqrt[4]{2}^{\;5-1}\left(\frac{1}{\sqrt[4]{3}}\right)^{n-5+1}$$

$$= {}^nC_{n-4}\; \sqrt[4]{2}^{\;4}\left(\frac{1}{\sqrt[4]{3}}\right)^{n-4}$$

$$= \frac{n!}{n-4\;![n-\; n-4\;]!}\times 2\times\frac{\sqrt[4]{3}^{\;4}}{\sqrt[4]{3}^{\;n}}$$

$$= \frac{n!}{n-4\;!4!}\;\frac{2\times 3}{\sqrt[4]{3}^{\;n}}$$

$$= \frac{n!}{4!\; n-4\;!}\;\frac{6}{\sqrt[4]{3}^{\;n}}$$

Given that,

$$\frac{5th\,term\ from\,beginning}{5th\,term\ from\,end} = \frac{\sqrt{6}}{1}$$

$$\Rightarrow \left(\frac{n!}{4!\,(n-4)!}\times\frac{(\sqrt[4]{2})^n}{6}\right) \div \left(\frac{n!}{4!\,(n-4)!}\times\frac{6}{(\sqrt[4]{3})^n}\right) = \sqrt{6}$$

$$\Rightarrow \frac{n!}{4!\,(n-4)!}\times\frac{(\sqrt[4]{2})^n}{6}\times\frac{4!\,(n-4)!}{n!}\times\frac{(\sqrt[4]{3})^n}{6} = \sqrt{6}$$

$$\Rightarrow (\sqrt[4]{2})^n\,(\sqrt[4]{3})^n = 36\sqrt{6}$$

$$\Rightarrow (\sqrt[4]{6})^n = 6^2\,.6^{1/2} \quad [\text{as } a^n b^n = (ab)^n]$$

$$\Rightarrow 6^{n/4} = 6^{5/2} \quad [\text{since, } a^n a^m = a^{n+m}]$$

So we get,

$$\frac{n}{4} = \frac{5}{2}$$

$$\Rightarrow n = \frac{5}{2}\times 4$$

$$\Rightarrow n = 10$$

Q9. Expand using Binomial Theorem $\left(1+\frac{x}{2}-\frac{2}{x}\right)^4$, $x \neq 0$.

A.9. $\left(1+\frac{x}{2}-\frac{2}{x}\right)^4$

Using binomial theorem we have,

$$\left[\left(1+\frac{x}{2}\right)-\frac{2}{x}\right]^4$$

$$= {}^4C_0\left(1+\frac{x}{2}\right)^4 + {}^4C_1\left(1+\frac{x}{2}\right)^3\left(-\frac{2}{x}\right) + {}^4C_2\left(1+\frac{x}{2}\right)^2\left(-\frac{2}{x}\right)^2 + {}^4C_3\left(1+\frac{x}{2}\right)\left(-\frac{2}{x}\right)^3 + {}^4C_4\left(-\frac{2}{x}\right)^4$$

$$= \left[\frac{4!}{0!\,(4-0)!}\left(1+\frac{x}{2}\right)^4\right] - \left[\frac{4!}{1!\,(4-1)!}\left(1+\frac{x}{2}\right)^3\left(\frac{2}{x}\right)\right] + \left[\frac{4!}{2!\,(4-2)!}\left(1+\frac{x}{2}\right)^2\left(\frac{4}{x^2}\right)\right] - \left[\frac{4!}{3!\,(4-3)!}\left(1+\frac{x}{2}\right)\left(\frac{8}{x^3}\right)\right] + \left[\frac{4!}{4!\,(4-4)!}\left(\frac{16}{x^4}\right)\right]$$

$$= 1\times\left(1+\frac{x}{2}\right)^4 - \left[\left(1+\frac{x}{2}\right)^3\left(\frac{2}{x}\right)\right] + \left[\frac{4\times 3\times 2!}{2\times 1\times 2!}\left(1+\frac{x}{2}\right)^2\left(\frac{4}{x^2}\right)\right] - \left[\frac{4\,x3!}{3!1!}\left(1+\frac{x}{2}\right)\left(\frac{8}{x^3}\right)\right] + \left[1\times\left(\frac{16}{x^4}\right)\right]$$

$$= \left(1+\frac{x}{2}\right)^4 - 4\left[\left(1+\frac{x}{2}\right)^3\left(\frac{2}{x}\right)\right] + 6\left[\left(1+\frac{x}{2}\right)^2\left(\frac{4}{x^2}\right)\right] - 4\left[\left(1+\frac{x}{2}\right)\left(\frac{8}{x^3}\right)\right] + \left(\frac{16}{x^4}\right) \quad \text{----- (1)}$$

Now,

$\left(1+\frac{x}{2}\right)^4$

$= {}^4C_0(1)^4 + {}^4C_1\,(1)^3\left(\frac{x}{2}\right) + {}^4C_2(1)^2\left(\frac{x}{2}\right)^2 + {}^4C_3(1)\left(\frac{x}{2}\right)^3 + {}^4C_4\left(\frac{x}{2}\right)^4$

$= \left(\frac{4!}{0!\,(4-0)!}\times 1\right) + \left(\frac{4!}{1!\,(4-1)!}\times 1\times\frac{x}{2}\right) + \left(\frac{4!}{2!\,(4-2)!}\times 1\times\frac{x^2}{4}\right) + \left(\frac{4!}{3!\,(4-3)!}\times 1\times\frac{x^3}{8}\right) +$
$\left(\frac{4!}{4!\,(4-4)!}\times 1\times\frac{x^4}{16}\right)$

$= 1 + \left(\frac{4\times 3!}{1\times 3!}\times\frac{x}{2}\right) + \left(\frac{4\times 3\times 2!}{2\times 1\times 2!}\times\frac{x^2}{4}\right) + \left(\frac{4\times 3!}{3!1!}\times\frac{x^3}{8}\right) + \left(1\times\frac{x^4}{16}\right)$

$= 1 + 2x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16}$ ------------------- (2)

$\left(1+\frac{x}{2}\right)^3$

$= {}^3C_0(1)^3 + {}^3C_1\,(1)^2\left(\frac{x}{2}\right) + {}^3C_2\,(1)\left(\frac{x}{2}\right)^2 + {}^3C_3\left(\frac{x}{2}\right)^3$

$= \left(\frac{3!}{0!\,(3-0)!}\times 1\right) + \left(\frac{3!}{1!\,(3-1)!}\times 1\times\frac{x}{2}\right) + \left(\frac{3!}{2!\,(3-2)!}\times 1\times\frac{x^2}{4}\right) + \left(\frac{3!}{3!\,(3-3)!}\times 1\times\frac{x^3}{8}\right)$

$= 1 + \left(\frac{3\times 2!}{1\times 2!}\times\frac{x}{2}\right) + \left(\frac{3\times 2!}{2!1!}\times\frac{x^2}{4}\right) + \left(1\times\frac{x^3}{8}\right)$

$= 1 + \frac{3x}{2} + \frac{3x^2}{4} + \frac{x^3}{8}$ ------------------------- (3)

And, $\left(1+\frac{x}{2}\right)^2$

$= {}^2C_0(1)^2 + {}^2C_1\,(1)\left(\frac{x}{2}\right) + {}^2C_2\left(\frac{x}{2}\right)^2$

$= \left(\frac{2!}{0!\,(2-0)!}\times 1\right) + \left(\frac{2!}{1!\,(2-1)!}\times 1\times\frac{x}{2}\right) + \left(\frac{2!}{2!\,(2-2)!}\times\frac{x2}{4}\right)$

$= 1 + \left(\frac{2\times 1}{1\times 1!}\times 1\times\frac{x}{2}\right) + \frac{x^2}{4}$

$= 1 + x + \frac{x^2}{4}$ ---------------------------- (4)

Putting (2), (3) and (4) in (1) we get,

$$\left(1+\frac{x}{2}-\frac{2}{x}\right)^4$$

$$= 1 + 2x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} - 4\left[\left(1+\frac{3x}{2}+\frac{3x^2}{2}+\frac{x^3}{2}\right)\left(\frac{2}{x}\right)\right] + 6\left[\left(1+x+\frac{x^2}{4}\right)\left(\frac{4}{x^2}\right)\right] - 4 \left[\left(1+\frac{x}{2}\right)\left(\frac{8}{x^3}\right)\right] + \left(\frac{16}{x^4}\right)$$

$$= 1 + 2x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} - \frac{8}{x} - 12 - 6x - x^2 + \frac{24}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} - \frac{16}{x^2} + \frac{16}{x^4}$$

$$= \frac{16}{x} + \frac{8}{x^2} - \frac{32}{x^3} + \frac{16}{x^4} - 4x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} - 5$$

Q10. Find the expansion of $(3x^2 - 2ax + 3a^2)^3$ using binomial theorem.

A.10. $\left[3x^2 - 2ax + 3a^2\right]^3$

$$= \left[3x^2 + a\ \ -2x + 3a\ \right]^3$$

We know that by binomial theorem,

$$(a+b)^3 = a^3 + b^3 + 3ab\ \ a+b$$

$$= a^3 + b^3 + 3a^2b + 3ab^2$$

Then,

$$\left[3x^2 + a\ \ -2x + 3a\ \right]^3$$

$$= (3x^2)^3 + \left[a\ \ -2x+3a\ \right]^3 + \left[3(3x^2)^2\, a\ \ -2x+3a\ \right] + \left[3(3x^2)\ \ a\ \ -2x+3a\ \ ^2\right]$$

$$= 27x^6 + \left[a^3\ \ -2x+3a\ \ ^3\right] + \left[3(9x^4)\ \ -2ax+3a^2\ \right] + \left[3(3x^2)\ \ a^2\ \ 3a-2x\ \ ^2\ \right]$$

$$= 27x^6 + \left[a^3\ \ -8x^3 + 27a^3 + 3\ \ 4x^2\ \ \ 3a\ \ +3\ \ -2x\ \ \ 9a^2\ \ \right] + \left[-54ax^5 + 81a^2x^4\right] + [\ \ 9a^2x^2\ \ 9a^2 + 4x^2 - 12ax\ \ ]$$

$$= 27x^6 + [-8a^3x^3 + 27a^6 + 36a^4x^2 - 54a^5x] + [-54ax^5 + 81a^2x^4] + [(81a^4x^2 + 36a^2x^4 - 108a^3x^3]$$

$$= 27x^6 - 8a^3x^3 + 27a^6 + 36a^4x^2 - 54a^5x - 54ax^5 + 81a^2x^4 + 81a^4x^2 + 36a^2x^4 - 108a^3x^3$$

$$= 27x^6 - 54ax^5 + 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6$$