Chapter 6
Systems of Particles and Rotational Motion

Rigid Body

Ideally a rigid body is a body with a perfectly definite and unchanging shape. The distances between different pairs of such a body do not change.

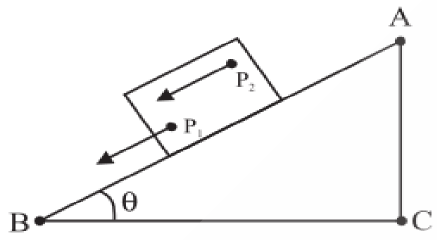
Motion of a rigid body

The motion of a rigid body which is not pivoted or fixed in some way is either a pure translation or a combination of translation and rotation.
The motion of a rigid body which is pivoted or fixed in some way is rotation.

1)Pure Translational Motion

In pure translational motion at any instant of time every particle of the body has the same velocity.

Eg: A block moving down an inclined plane.

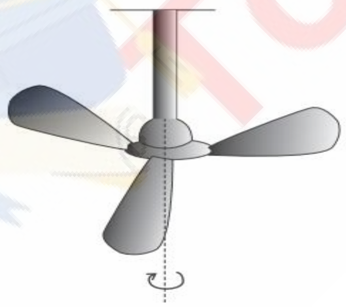


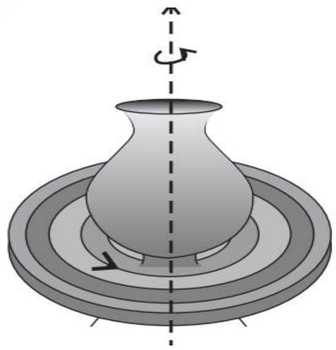
Any point like P1 or P2 of the block moves with the same velocity at any instant of time.

2)Pure Rotational Motion

In pure rotational motion at any instant of time every point in the rotating rigid body has the same angular velocity,but different linear velocity.

i) Rotation about a fixed axis



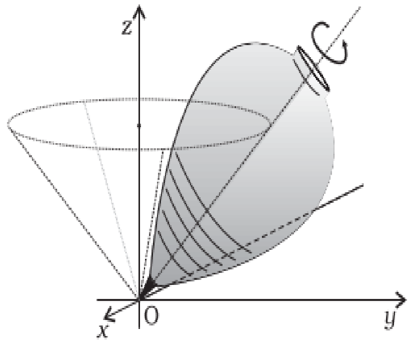


Eg: A ceiling fan   A potter's wheel.

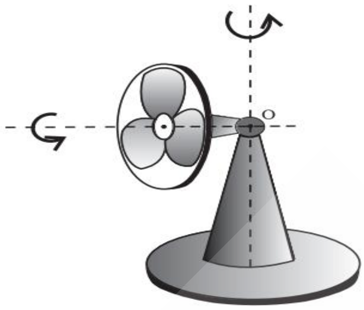
The line along which the body is fixed is termed as its axis of rotation.

In rotation of a rigid body about a fixed axis, every particle of the body moves in a circle, which lies in a plane perpendicular to the axis and has its centre on the axis.

ii) Rotation about an axis which is not fixed



Eg: A spinning top

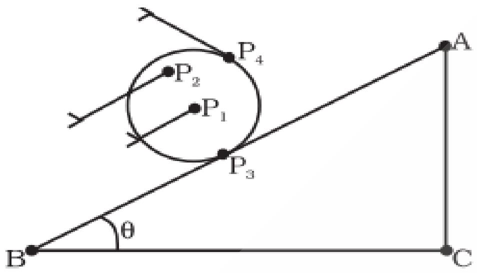


An oscillating table fan

3)Rolling Motion

It is a combination of translational and rotational motion.

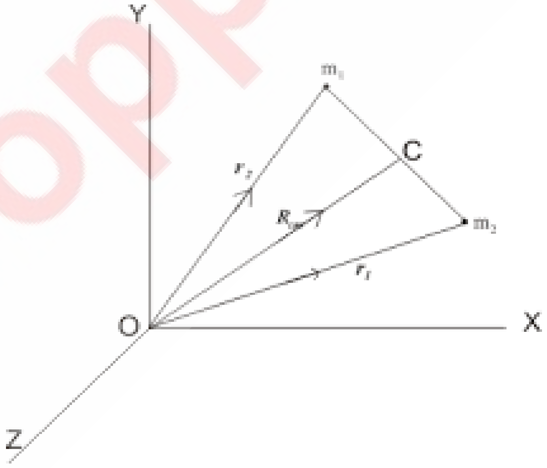
Eg A solid cylinder moving down an inclined plane.



Points P1, P2, P3 and P4 have different velocities at any instant of time. In fact, the velocity of the point of contact P3 is zero at any instant, if the cylinder rolls without slipping.

Centre Of Mass

The centre of is a hypothetical point where the entire mass of an object may be assumed to be concentrated to visualise its motion.



Consider a two particle system. Let C be the centre of mass which is at a distancev X from origin.

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M} \quad \text{where } M = m_1 + m_2$$

x coordinate of centre of mass $X = \frac{m_1x_1+m_2x_2}{m_1+m_2}$

y coordinate of centre of mass $Y = \frac{m_1y_1+m_2y_2}{m_1+m_2}$

z coordinate of centre of mass $Z = \frac{m_1z_1+m_2z_2}{m_1+m_2}$

If we have n particles of masses $m_1, m_2, \dots m_n$

$$\vec{R} = \frac{m_1\vec{r}_1+m_2\vec{r}_2+\cdots\dots\dots+m_n\vec{r}_n}{M} \qquad \text{--------(1)}$$

$$\vec{R} = \frac{\sum m_i\vec{r}_i}{M} \qquad \text{where } M = m_1 + m_2 + \dots\dots + m_n$$

If the origin is chosen to be the centre of mass then $\vec{R} = 0$

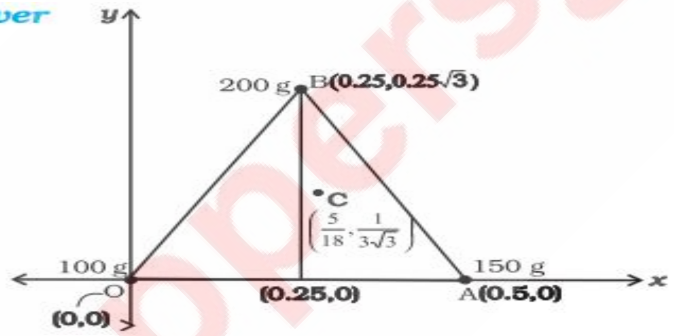
$$0 = \frac{\sum m_i\vec{r}_i}{M}$$

$$\sum m_i\vec{r}_i = 0$$

Example

Find the centre of mass of three particles at the vertices of an equilateral triangle. The masses of the particles are 100g, 150g, and 200g respectively. Each side of the equilateral triangle is 0.5m long.

Answer



$$X = \frac{m_1x_1+m_2x_2+m_3x_3}{m_1+m_2+m_3}$$

$$= \frac{[100(0)+150(0.5)+200(0.25)]\ \text{g m}}{(100+150+200)\ \text{g}}$$

$$= \frac{75+50}{450}\text{m} = \frac{125}{450}\text{m} = \frac{5}{18}\text{m}$$

$$Y = \frac{m_1y_1+m_2y_2+m_3y_3}{m_1+m_2+m_3}$$

$$Y = \frac{[100(0)+150(0)+200(0.25\sqrt{3})]\ \text{g m}}{450\ \text{g}}$$

$$= \frac{50\sqrt{3}}{450}\text{m} = \frac{\sqrt{3}}{9}\text{m} = \frac{1}{3\sqrt{3}}\text{m}$$

Motion of Centre of Mass

Position vector of centre of mass

$$\vec{R} = \frac{\sum m_i \vec{r}_i}{M}$$

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots \ldots\ldots + m_n\vec{r}_n}{M} \qquad \text{--------(1)}$$

where $M = m_1 + m_2 + \ldots\ldots + m_n$

Velocity of centre of mass

Differentiating

$$\frac{d}{dt}\vec{R} = \frac{d}{dt}\left\{\frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots \ldots\ldots + m_n\vec{r}_n}{M}\right\}$$

$$\vec{V} = \frac{m_1 \frac{d}{dt}\vec{r}_1 + m_2 \frac{d}{dt}\vec{r}_2 + \cdots \ldots .. + m_n \frac{d}{dt}\vec{r}_n}{M}$$

$$\vec{V} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots \ldots\ldots + m_n\vec{v}_n}{M} \qquad \text{-------------(2)}$$

Acceleration of centre of mass

Differentiating

$$\frac{d}{dt}\vec{V} = \frac{d}{dt}\left\{\frac{m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots \ldots\ldots + m_n\vec{v}_n}{M}\right\}$$

$$\vec{A} = \frac{m_1 \frac{d}{dt}\vec{v}_1 + m_2 \frac{d}{dt}\vec{v}_2 + \cdots \ldots .. + m_n \frac{d}{dt}\vec{v}_n}{M}$$

$$\vec{A} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots \ldots\ldots + m_n\vec{a}_n}{M} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Force on centre of mass

Acceleration of centre of mass

$$\vec{A} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots \ldots \ldots .. + m_n\vec{a}_n}{M}$$

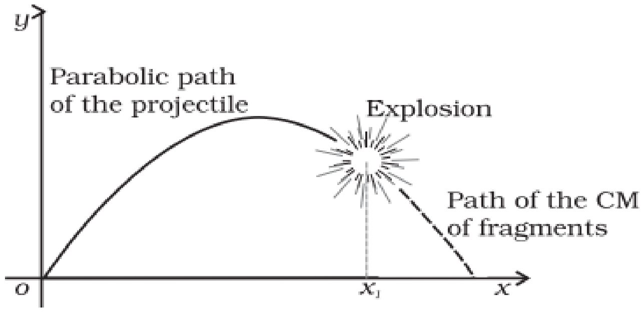
$$M\vec{A} = m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots \ldots \ldots .. + m_n\vec{a}_n$$

$$\vec{F}_{ext} = m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots \ldots \ldots .. + m_n\vec{a}_n$$

$$\vec{F}_{ext} = \vec{F}_1 + \vec{F}_2 + \cdots \ldots \ldots .. + \vec{F}_n$$

$$\vec{F}_{ext} = M\vec{A}$$

The centre of mass of a system of particles moves as if all the mass of the system was concentrated at the centre of mass and all the external forces were applied at that point.



The centre of mass of the fragments of the projectile continues along the same parabolic path which it would have followed if there were no explosion.

Linear Momentum of centre of mass

Velocity of centre of mass

$$\vec{V} = \frac{m_1\vec{v}_1+m_2\vec{v}_2+\cdots\ldots\ldots+m_n\vec{v}_n}{M}$$

$$M\vec{V} = m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots\ldots\ldots+m_n\vec{v}_n$$

$$\vec{P} = \vec{p}_1 + \vec{p}_2 + \cdots\ldots\ldots+\vec{p}_n$$

Law of Conservation of Momentum for a System of Particles

If Newton's second law is extended to a system of particles,

$$\vec{F}_{ext} = \frac{d\vec{P}}{dt}$$

When the sum of external forces acting on a system of particles is zero

$$\vec{F}_{ext} = 0$$

$$\frac{d\vec{P}}{dt} = 0$$

$$\vec{P} = \text{constant}$$

Thus, when the total external force acting on a system of particles is zero, the total linear momentum of the system is constant. This is the law of conservation of the total linear momentum of a system of particles.

$$\text{But } \vec{P} = M\vec{V}$$

$$M\vec{V} = \text{constant}$$

$$\vec{V} = constant$$

When the total external force on the system is zero the velocity of the centre of mass remains constant or the CM of the system is in uniform motion.

Vector Product or Cross product of Two Vectors

Vector product of two vectors $\vec{A}$ and $\vec{B}$ is defined as

$$\vec{A} \times \vec{B} = AB \sin\theta \, \hat{n}$$

where A and B are magnitudes of $\vec{A}$ and $\vec{B}$

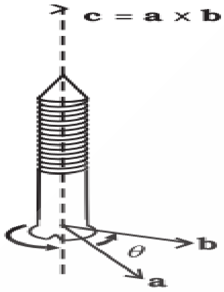
θ is the angle between $\vec{A}$ and $\vec{B}$

$\hat{n}$ is the unit vector perpendicular to the plane containing $\vec{A}$ and $\vec{B}$

The direction of $\vec{A} \times \vec{B}$ is given by right hand screw rule or right hand rule

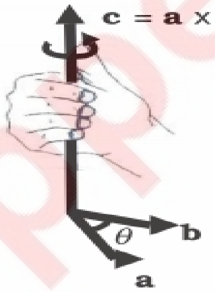
Right hand screw rule

If we turn the head of screw in the direction from $\vec{A}$ to $\vec{B}$,then the tip of the screw advances in the direction of $\vec{A} \times \vec{B}$



Right hand rule

if the fingers of right hand are curled up in the direction from $\vec{A}$ to $\vec{B}$, then the stretched thumb points in the direction of $\vec{A} \times \vec{B}$



- The vector product is not commutative

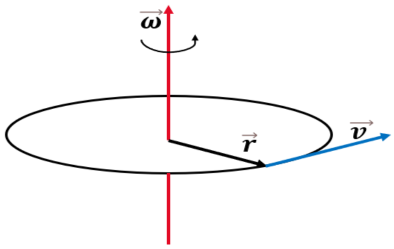
$$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$$

- Vector product obeys distributive law

$$\vec{A} \times (\overrightarrow{B + C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

- $\vec{A} \times \vec{A} = \vec{0}$
- $\hat{\imath} \times \hat{\imath} = 0, \quad \hat{\jmath} \times \hat{\jmath} = 0, \quad \hat{k} \times \hat{k} = 0$
- $\hat{\imath} \times \hat{\jmath} = \hat{k}, \quad \hat{\jmath} \times \hat{k} = \hat{\imath}, \quad \hat{k} \times \hat{\imath} = \hat{\jmath}$
- $\hat{\jmath} \times \hat{\imath} = -\hat{k}, \quad \hat{k} \times \hat{\jmath} = -\hat{\imath}, \quad \hat{\imath} \times \hat{k} = -\hat{\jmath}$

Angular Velocity and its Relation with Linear Velocity



The angular velocity is a vector quantity. $\vec{\omega}$ is directed along the fixed axis as shown.

The linear velocity of the particle is

$$\vec{v} = \vec{\omega} \times \vec{r}$$

It is perpendicular to both $\vec{\omega}$ and $\vec{r}$ and is directed along the tangent to the circle described by the particle.

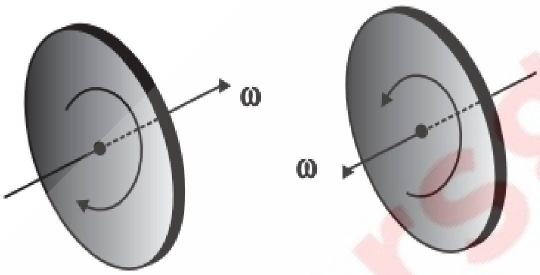


Figure shows the direction of angular velocity when the body rotates in clockwise and anti clockwise direction.

For rotation about a fixed axis, the direction of the vector ω does not change with time. Its magnitude may change from instant to instant. For the more general rotation, both the magnitude and the direction of ω may change from instant to instant.

Angular acceleration

Angular acceleration $\vec{\alpha}$ is defined as the time rate of change of angular velocity.

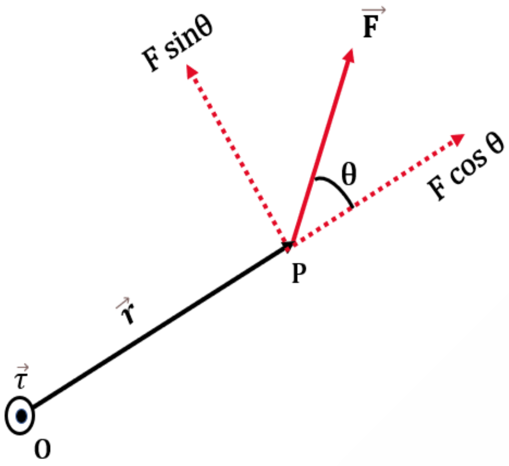
$$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$$

If the axis of rotation is fixed, the direction of ω and hence, that of α is fixed. In this case the vector equation reduces to a scalar equation

$$\alpha = \frac{d\omega}{dt}$$

Torque or Moment of Force

The rotational analogue of force is torque or moment of force .



If a force $\vec{\mathbf{F}}$ acts on a single particle at a point P whose position with respect to the origin O is $\vec{r}$,then torque about origin o is

$$\vec{\tau} = r\,F\,\sin\theta$$

$$\vec{\tau} = \vec{r} \times \vec{F}$$

- Torque has dimensions $M\,L^2T^{-2}$
- Its dimensions are the same as those of work or energy.
- It is a very different physical quantity than work.
- Moment of a force is a vector, while work is a scalar.
- The SI unit of moment of force is Newton-metre (Nm)

The magnitude of the moment of force may be written

$$\tau = (r \sin\theta)\,F = r_{\perp}\,F$$

$$\tau = r\,(F \sin\theta) = r\,F_{\perp}$$

where $r_{\perp} = r \sin\theta$ is the perpendicular distance of the line of action of F form the origin and $F_{\perp} = F\sin\theta$ is the component of F in the direction perpendicular to r.

Angular momentum of a particle

Angular momentum is the rotational analogue of linear momentum.

Angular momentum is a vector quantity. It could also be referred to as moment of (linear) momentum.

$$\vec{l} = \vec{r} \times \vec{p}$$

$$\vec{l} = rp\sin\theta$$

Relation connecting Torque and Angular momentum

$$\vec{l} = \vec{r} \times \vec{p}$$

Differentiating

$$\frac{d\vec{l}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p})$$

$$\frac{d\vec{l}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$\vec{p} = m\vec{v}, \quad \frac{d\vec{r}}{dt} = \vec{v}, \quad \frac{d\vec{p}}{dt} = \vec{F}$$

$$\frac{d\vec{l}}{dt} = \vec{v} \times m\vec{v} + \vec{r} \times \vec{F}$$

$$\vec{v} \times \vec{v} = 0, \quad (\vec{r} \times \vec{F} = \vec{\tau})$$

$$\frac{d\vec{l}}{dt} = 0 + \vec{\tau}$$

$$\vec{\tau} = \frac{d\vec{l}}{dt}$$

Thus, the time rate of change of the angular momentum of a particle is equal to the torque acting on it.

This is the rotational analogue of the equation $\vec{F} = \frac{d\vec{p}}{dt}$, which expresses Newton's second law for the translational motion of a single particle.

Relation connecting Torque and Angular momentum for a system of particles

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

where $\vec{L} = \vec{l}_1 + \vec{l}_2 + \cdots + \vec{l}_n$

Law of Conservation of Angular momentum

For a system of particles

$$\vec{\tau}_{ext} = \frac{d\vec{L}}{dt}$$

If external torque $\vec{\tau}_{ext} = 0$,

$$\frac{d\vec{L}}{dt} = 0$$

$$\vec{L} = \text{constant}$$

If the total external torque on a system of particles is zero, then the total angular momentum of the system is conserved i.e, remains constant.

Example

Find the torque of a force $7\hat{i} + 3\hat{j} - 5\hat{k}$ about the origin. The force acts on a particle whose position vector is $\hat{i} - \hat{j} + \hat{k}$.

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\vec{\tau} = (\hat{i} - \hat{j} + \hat{k}) \times (7\hat{i} + 3\hat{j} - 5\hat{k})$$

$$\begin{matrix} + & - & + \end{matrix}$$

$$\vec{\tau} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 7 & 3 & -5 \end{vmatrix}$$

$$\vec{\tau} = \hat{i}\,[(-1 \times -5) - (3 \times 1)] - \hat{j}\,[(1 \times -5) - (7 \times 1)] + \hat{k}\,[(1 \times 3) - (7 \times -1)]$$

$$\vec{\tau} = \hat{i}\,[5 - 3] - \hat{j}\,[-5 - 7] + \hat{k}\,[3 - -7]$$

$$\vec{\tau} = 2\hat{i} + 12\hat{j} + 10\hat{k}$$

Equilibrium of a Rigid Body

A rigid body is said to be in mechanical equilibrium, if it is in both translational equilibrium and rotational equilibrium.

i.e, for a body in mechanical equilibrium its linear momentum and angular momentum are not changing with time.

Translational Equilibrium

When the total external force on the rigid body is zero, then the total linear momentum of the body does not change with time and the body will be in translational equilibrium .

$$\mathbf{F}_1 + \mathbf{F}_2 + \ldots + \mathbf{F}_n = \sum_{i=1}^{n} \mathbf{F}_i = \mathbf{0}$$

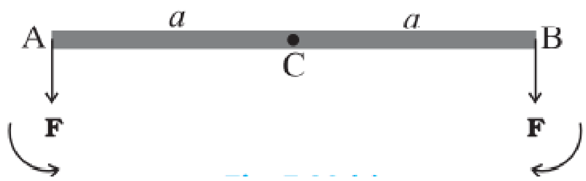
Rotational Equilibrium

When the total external torque on the rigid body is zero, the total angular momentum of the body does not change with time and the body will be in rotational equilibrium .

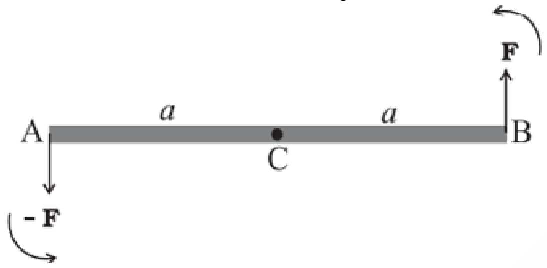
$$\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 + \ldots + \boldsymbol{\tau}_n = \sum_{i=1}^{n} \boldsymbol{\tau}_i = \mathbf{0}$$

Partial equilibrium

A body may be in partial equilibrium, i.e., it may be in translational equilibrium and not in rotational equilibrium, or it may be in rotational equilibrium and not in translational equilibrium.



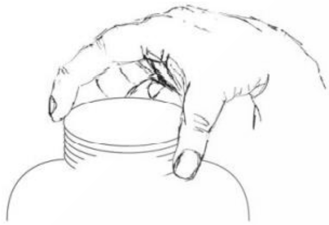
Here net torque is zero and the body is in rotational equilibrium.
Net force is not zero and the body is not in traslational equilibrium



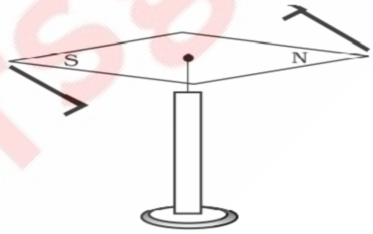
Here net torque is not zero and the body will not be rotational equilibrium.
Net force is zero and the body will be in traslational equilibrium.

Couple

A pair of equal and opposite forces with different lines of action is known as a couple. A couple produces rotation without translation.

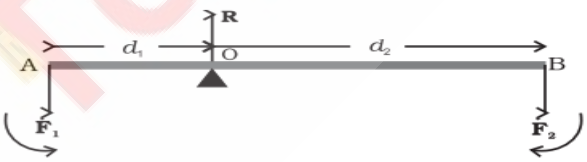


Our fingers apply a couple to turn the lid



The Earth's magnetic field exerts equal and opposite forces on the poles of a compass needle. These two forces form a couple.

Principles of Moments



The lever is a system in mechanical equilibrium.
For rotational equilibrium the sum of moments must be zero,

$$\mathbf{d_1F_1 - d_2F_2 = 0}$$

The equation for the principle of moments for a lever is

$$\mathbf{d_1F_1 = d_2F_2}$$

load arm × load = effort arm × effort

$$\text{Mechanical Advantage MA} = \frac{F_1}{F_2} = \frac{d_2}{d_1}$$

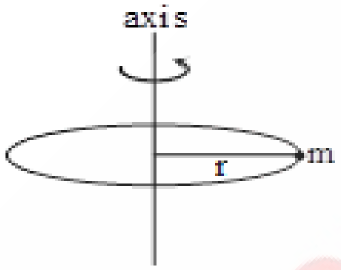
Centre of gravity

The Centre of gravity of a body is the point where the total gravitational torque on the body is zero.

- The centre of gravity of the body coincides with the centre of mass. For a body is small, g does not vary from one point of the body to the other. Then the centre of gravity of the body coincides with the centre of mass.
- If the body is so extended that g varies from part to part of the body, then the centre of gravity and centre of mass will not coincide.

Moment of Inertia

Moment of Inertia is the rotational analogue of mass.
Moment of inertia is a measure of rotational inertia



The moment of inertia of a particle of mass m rotating about an axis is

$$I = mr^2$$

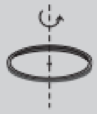
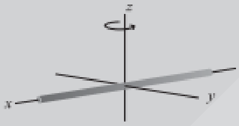
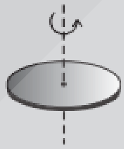
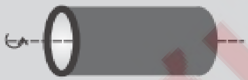
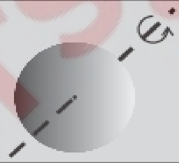
The moment of inertia of a rigid body is

$$I = \sum_{i=1}^{n} m_i r_i^2$$

The moment of inertia of a rigid body depends on

- The mass of the body, its shape and size
- Distribution of mass about the axis of rotation
- The position and orientation of the axis of rotation.

Moments of Inertia of some regular shaped bodies about specific axes

Z	Body	Axis	Figure	I
(1)	Thin circular ring, radius *R*	Perpendicular to plane, at centre		MR^2
(2)	Thin circular ring, radius *R*	Diameter		$MR^2/2$
(3)	Thin rod, length L	Perpendicular to rod, at mid point		$ML^2/12$
(4)	Circular disc, radius *R*	Perpendicular to disc at centre		$MR^2/2$
(5)	Circular disc, radius *R*	Diameter		$MR^2/4$
(6)	Hollow cylinder, radius *R*	Axis of cylinder		MR^2
(7)	Solid cylinder, radius *R*	Axis of cylinder		$MR^2/2$
(8)	Solid sphere, radius *R*	Diameter		$2MR^2/5$

Rotational Kinetic energy

Cosider a particle of mass m rotating about an axis of radius r with angular velocity ω

The kinetic energy of motion of this particle is

$$kE = \frac{1}{2}mv^2$$

But $v = r\,\omega$

$$kE = \frac{1}{2}mr^2\omega^2$$

$I = mr^2$

$$\textbf{Rotational kE} = \frac{1}{2}I\omega^2$$

Radius of Gyration (k)

The radius of gyration can be defined as the distance of a mass point from the axis of roatation whose mass is equal to the whole mass of the body and whose moment of inertia is equal to moment of inertia of the whole body about the axis.

If K is the radius of gyration, we can write

$$I = Mk^2$$

$$k = \sqrt{\frac{I}{M}}$$

Flywheel

The machines, such as steam engine and the automobile engine, etc., that produce rotational motion have a disc with a large moment of inertia, called a flywheel. Because of its large moment of inertia, the flywheel resists the sudden increase or decrease of the speed of the vehicle. It allows a gradual change in the speed and prevents jerky motions, thereby ensuring a smooth ride for the passengers on the vehicle.

Kinematics of Rotational Motion about a Fixed Axis

The kinematical equations of linear motion with uniform (i.e. constant) acceleration

$$v = v_0 + at$$

$$x = x_0 + v_0 t + \frac{1}{2} a t^2$$

$$v^2 = v_0^2 + 2ax$$

The corresponding kinematic equations for rotational motion with uniform angular acceleration are:

$$\omega = \omega_0 + \alpha t$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2$$

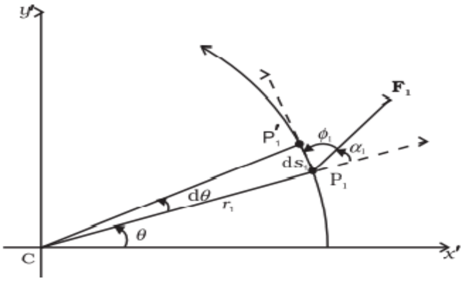
$$\text{and } \omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

Dynamics of Rotational Motion about a Fixed Axis

Comparison of Translational and Rotational Motion

	Linear Motion	Rotational Motion about a Fixed Axis
1	Displacement x	Angular displacement θ
2	Velocity $v = dx/dt$	Angular velocity $\omega = d\theta/dt$
3	Acceleration $a = dv/dt$	Angular acceleration $\alpha = d\omega/dt$
4	Mass M	Moment of inertia I
5	Force $F = Ma$	Torque $\tau = I\alpha$
6	Work $dW = F\,ds$	Work $W = \tau\, d\theta$
7	Kinetic energy $K = Mv^2/2$	Kinetic energy $K = I\omega^2/2$
8	Power $P = F v$	Power $P = \tau\omega$
9	Linear momentum $p = Mv$	Angular momentum $L = I\omega$

Work done by a torque



Work done by a force F acting on a particle of a body rotating about a fixed axis

$$dW = F.\,ds$$

$$dW = F ds \cos\varphi$$

but $\varphi + \alpha = 90$, $\varphi = 90 - \alpha$

$\cos(90 - \alpha) = \sin\alpha$

$$dW = F(r\,d\theta)\sin\alpha$$

$$dW = r\,F\sin\alpha\,d\theta$$

$$dW = \tau\,d\theta$$

$$\mathbf{W = \tau\,\theta}$$

Instantaneous power by a Torque

$$P = \frac{dW}{dt}$$

But, $dW = \tau\,d\theta$

$$P = \tau\frac{d\theta}{dt}$$

$$\mathbf{P = \tau\omega}$$

Angular Momentum in Case of Rotation about a Fixed Axis.(or) Relation Connecting Angular Momentum And Moment Of Inertia

Angular momentum of a particle, $\vec{l} = \vec{r} \times \vec{p}$

For a system of particles , $\vec{L} = \sum \vec{l}$

$$\vec{L} = \sum \vec{r} \times \vec{p}$$

$$\vec{L} = \sum rp\sin 90\,\hat{k}$$

$$\vec{L} = \sum rp\,\hat{k}$$

$$\vec{L} = \sum rmv\,\hat{k} \qquad (p=mv)$$

$$\vec{L} = \sum rm(r\omega)\,\hat{k} \qquad (v=r\omega)$$

$$\vec{L} = \sum mr^2\omega\,\hat{k}$$

$$\vec{L} = I\omega\hat{k}$$

$$\vec{\mathbf{L}} = \mathbf{I}\vec{\boldsymbol{\omega}}$$

Relation Connecting Torque and Angular Acceleration

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

$$\text{But } \vec{L} = I\vec{\omega}$$

$$\vec{\tau} = \frac{dI\vec{\omega}}{dt}$$

$$\vec{\tau} = I\,\frac{d\vec{\omega}}{dt}$$

$$\vec{\tau} = I\,\vec{\alpha}$$

Conservation of angular momentum

If the external torque is zero, angular momentum is constant.

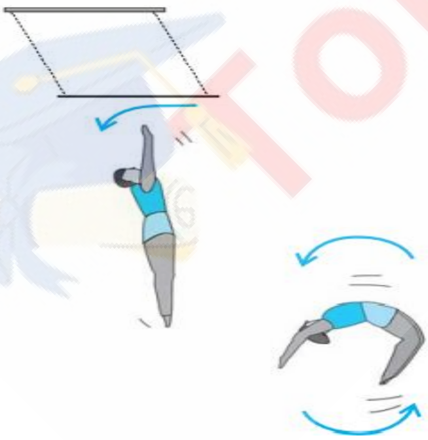
$$L = \text{constant}$$

$$I\omega = \text{constant}$$

When I increases ,ω decreases and vice versa, so that Iω is constant.

While the chair is rotating with considerable angular speed , if you stretch your arms horizontally, moment of inertia(I) increases and as a result, the angular speed(ω) is reduced.

If you bring back your arms closer to your body, moment of inertia(I) decreases and as a result, the angular speed(ω) increases again.



A circus acrobat and a diver take advantage of this principle.

Also, skaters and classical, Indian or western, dancers performing a pirouette on the toes of one foot display 'mastery' over this principle.