Chapter 3
Motion in a Plane

3.1 Introduction

In one dimension only two directions are possible and we used positive and negative signs to represent the two directions. But in order to describe motion of an object in two dimensions (plane) or three dimensions (space) we need to use vectors to describe physical quantities like position, displacement, velocity ,acceleration etc

Scalars and Vectors

A scalar quantity has only magnitude and no direction. It is specified completely by a single number, along with the proper unit.

Eg. distance ,mass , temperature, time .

A vector quantity has both magnitude and direction and obeys the triangle law of addition or the parallelogram law of addition. A vector is specified by giving its magnitude by a number and its direction.

Eg.displacement, velocity, acceleration and force.

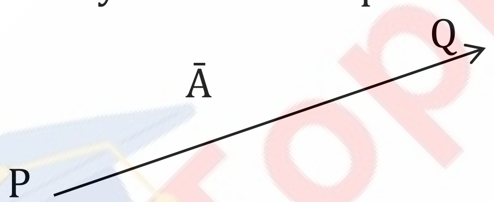
Representation of a Vector

A vector is representedby a bold letter say **A** or an arrow by an arrow placed over a letter, say $\bar{A}$.

The magnitude of a vector is called its absolute value, indicated by

$$|\bar{A}| = A$$

Graphically a vector is represented by a line segment with an arrow head.

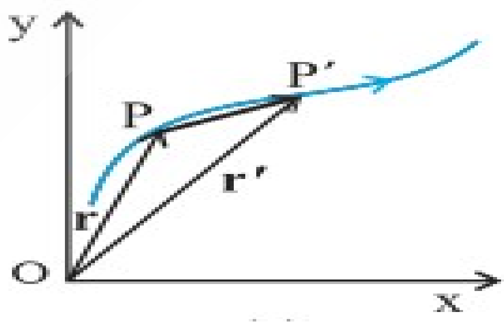


Q is the head of the vector

P is the tail of the vector

The length of line segment gives the magnitude of the vector and arrow mark gives its direction.

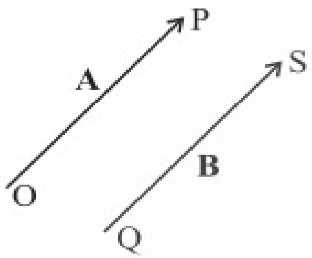
Position and Displacement Vectors



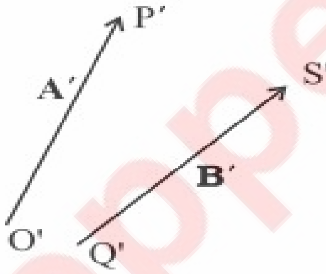
Let P and P′ be the positions of the object at time t and t′ , respectively .
OP is the position vector of the object at time t. OP = r.
OP’ is the position vector of the object at time t’. OP’= r’
If the object moves from P to P′ , the vector PP′ is called the displacement vector.
Displacement vector is the straight line joining the initial and final positions and does not depend on the actual path undertaken by the object between the two positions.

Equality of Vectors

Two vectors A and B are said to be equal if, and only if, they have the same magnitude and the same direction.
(a) Two equal vectors **A** and **B**.



(b) Two vectors **A** ′ and **B** ′ are unequal eventhough they are of same length

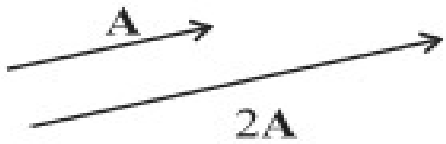


3.3 Multiplication of Vectors by Real Numbers

- Multiplying a vector $\bar{A}$ with a positive number λ gives a vector whose magnitude is changed by the factor λ but direction is the same as that of $\bar{A}$

$$\lambda \times \bar{A} = \lambda \bar{A}, \qquad \text{if } \lambda > 0$$

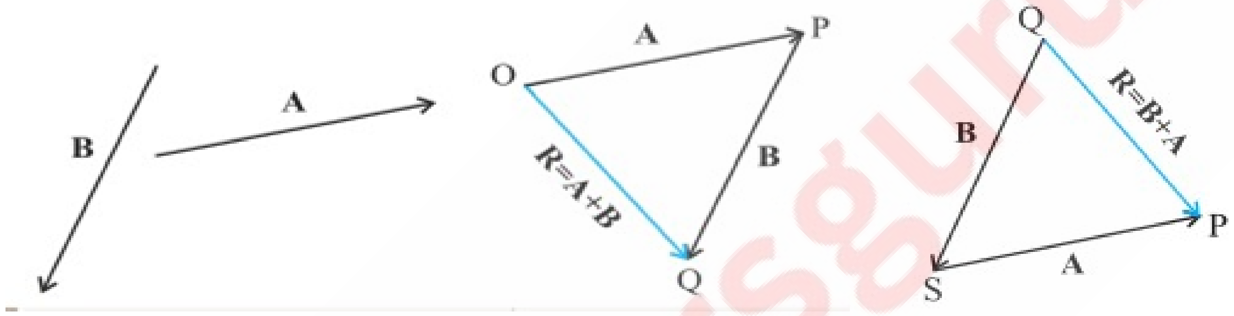
For example, if $\bar{A}$ is multiplied by 2, the resultant vector 2 $\bar{A}$ is in the same direction as $\bar{A}$ and has a magnitude twice of | $\bar{A}$



- Multiplying a vector $\bar{A}$ by a negative number λ gives a vector $\lambda\,\bar{A}$ whose direction is opposite to the direction of $\bar{A}$ and whose magnitude is
  $-\lambda$ times $|\bar{A}|$.
  For example, multiplying a given vector A by negative numbers, say -1 and -1.5, gives vectors as

3.4 Addition and Subtraction of Vectors — Graphical Method

Triangle law of vector addition



If two vectors are represented in magnitude and direction by the two sides of a triangle taken in order ,then their resultant is given by the third side of the triangle taken in reverse order.
This graphical method is called the head-to-tail method.
If we find the resultant of B + A , the same vector R is obtained.

- Thus, vector addition is commutative:

$$A + B = B + A$$

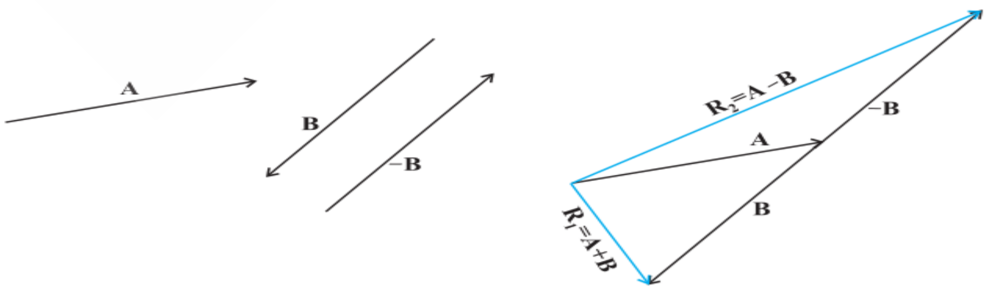
- The addition of vectors also obeys the associative law

$$(A + B) + C = A + (B + C)$$

Subtraction of vectors

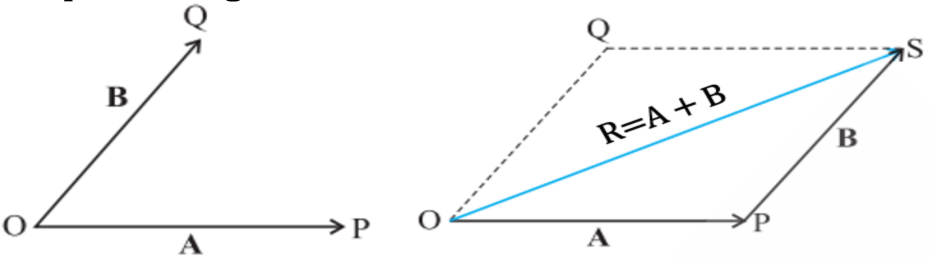
Subtraction of vectors can be defined in terms of addition of vectors. We define the difference of two vectors A and B as the sum of two vectors A and –B :

$$A - B = A + (-B)$$



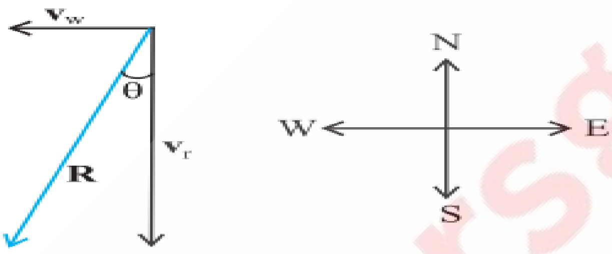
Parallelogram law of vector addition

If two vectors are represented in magnitude and direction by the adjacent sides of a parallelogram ,then their resultant is given by the diagonal of the parallelogram.



Example

Rain is falling vertically with a speed of 35 m s $^{-1}$. Winds starts blowing after sometime with a speed of 12 m s $^{-1}$ in east to west direction. In which direction
should a boy waiting at a bus stop hold his umbrella ?



$$R = \sqrt{v_r^2 + v_w^2} = \sqrt{35^2 + 12^2} \text{ m s}^{-1} = 37 \text{ m s}^{-1}$$

The direction θ that R makes with the vertical is given by

$$\tan\theta = \frac{v_w}{v_r} = \frac{12}{35} = 0.343$$

$$\theta = \tan^{-1}(0.343) = 19°$$

Therefore, the boy should hold his umbrella in the vertical plane at an angle of about 19° with the vertical towards the east.

Unit vectors

A unit vector is a vector of unit magnitude and points in a particular direction.
It has no dimension and unit. It is used to specify a direction only.
If we multiply a unit vector, say n by a scalar, the result is a vector.

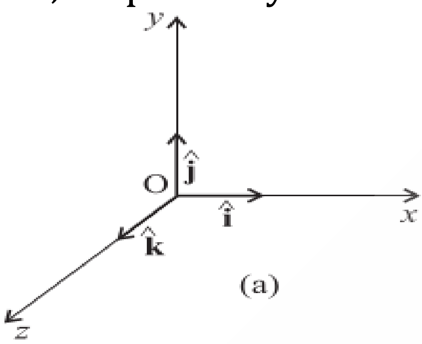
In general, a vector A can be written as

$$\bar{A} = |\bar{A}|\hat{A}$$

$$\hat{\mathbf{A}} = \frac{\bar{\mathbf{A}}}{|\bar{A}|}$$

where $\hat{A}$ is the unit vector along $\bar{A}$

Unit vectors along the x-, y- and z-axes of a rectangular coordinate system are denoted by $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$, respectively.



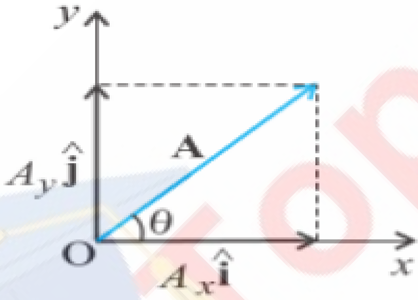
(a)

Since these are unit vectors, we have $|\hat{\imath}| = |\hat{\jmath}| = |\hat{k}| = 1$

These unit vectors are perpendicular to each other and are called orthogonal unit vectors

3.5 Resolution of a vector

We can now resolve a vector A in terms of component vectors that lie along unit vectors $\hat{\imath}$ and $\hat{\jmath}$.



$$\bar{A} = A_x\,\hat{\imath} + A_y\,\hat{\jmath}$$

The quantities A_x and A_y are called x-, and y- components of the vector A.

Note that A_x and A_y are not vectors, but $A_x\hat{\imath}$ and $A_y\hat{\jmath}$ are vectors.

Using simple trigonometry, we can express A_x and A_y in terms of the magnitude of A and the angle θ it makes with the x-axis :

$$\cos\theta = \frac{A_x}{A}$$

$$\mathbf{A_x = A\cos\theta}$$

$$\sin\theta = \frac{A_y}{A}$$

$$\mathbf{A_y = A\sin\theta}$$

$$A_x^2 + A_y^2 = A^2\cos^2\theta + A^2\sin^2\theta = A^2$$

$$\mathbf{A} = \sqrt{\mathbf{A_x^2 + A_y^2}}$$

And $\tan\theta = \frac{A_y}{A_x}, \quad \theta = \tan^{-1}\frac{A_y}{A_x}$

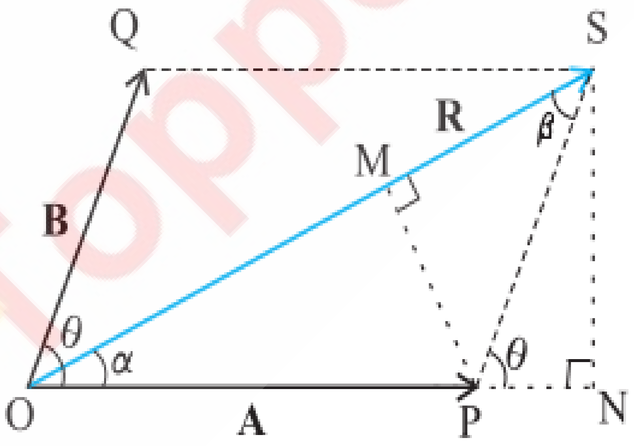
In general, if $\bar{A} = A_x\,\hat{\imath} + A_y\,\hat{\jmath} + A_y\,\hat{k}$

The magnitude of vector A is

$$\mathbf{A} = \sqrt{\mathbf{A_x^2 + A_y^2 + A_z^2}}$$

3.6 Vector Addition – Analytical Method

Consider two vectors **A** and **B** in x-y plane



SN is normal to OP and PM is normal to OS.

△SNP , $\cos\theta = PN / PS$ $\qquad$ $\sin\theta = SN / PS$

$\cos\theta = PN / B$ $\qquad$ $\sin\theta = SN / B$

$PN = B\cos\theta$ $\qquad$ $SN = B\sin\theta$

From the geometry of the figure,

$$OS^2 = ON^2 + SN^2$$

$$\text{but } ON = OP + PN$$

$$= A + B\cos\theta$$

$$SN = B\sin\theta$$

$$OS^2 = (A + B\cos\theta)^2 + (B\sin\theta)^2$$

$$R^2 = A^2 + 2AB\cos\theta + B^2\cos^2\theta + B^2\sin^2\theta$$

$$R^2 = A^2 + B^2 + 2AB\cos\theta$$

$$\mathbf{R = \sqrt{A^2 + B^2 + 2AB\cos\theta}}$$

This Equation gives the magnitude of the resultant of vectors A and B.
From figure,

$$\frac{R}{\sin\theta} = \frac{B}{\sin\alpha}$$

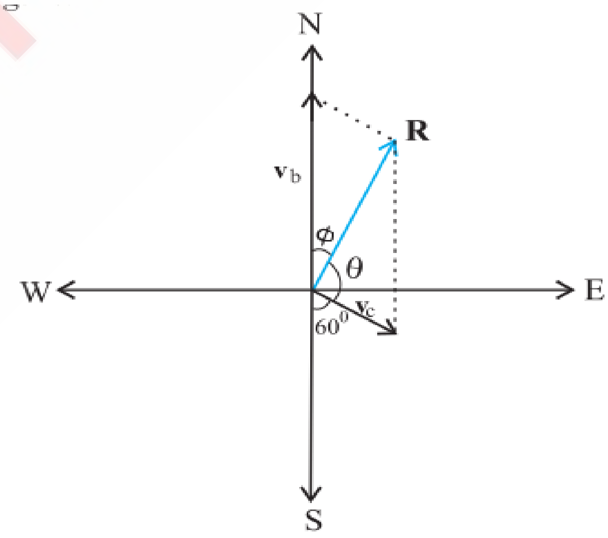
$$\sin\alpha = \frac{B}{R}\sin\theta$$

$$\tan\alpha = \frac{SN}{OP + PN} = \frac{B\sin\theta}{A + B\cos\theta}$$

These Equations gives the direction of the resultant of vectors A and B.

Example

A motorboat is racing towards north at 25 km/h and the water current in that region is 10 km/h in the direction of 60° east of south. Find the resultant velocity of the boat



$$R=\sqrt{v_b^2+v_c^2+2v_b v_c \cos 120^\circ}$$

$$=\sqrt{25^2+10^2+2\times25\times10(-1/2)}\cong 22\ \text{km/h}$$

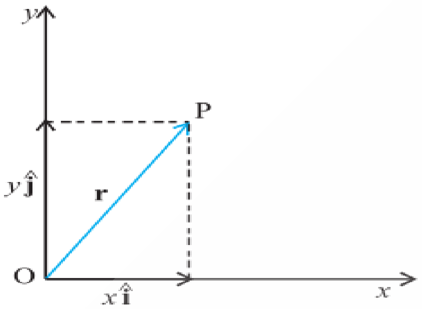
To obtain the direction, we apply the Law of sines

$$\frac{R}{\sin\theta}=\frac{v_c}{\sin\phi} \quad \text{or, } \sin\phi=\frac{v_c}{R}\sin\theta$$

$$=\frac{10\times\sin 120^\circ}{21.8}=\frac{10\sqrt{3}}{2\times21.8}\cong 0.397$$

$$\phi\cong 23.4^\circ$$

3.7 Motion in a Plane



Position Vector

The position vector r of a particle P at time t

$$\mathbf{r} = x\hat{i} + y\hat{j}$$

The position vector r of a particle P at time t'

$$\mathbf{r}' = x'\hat{i} + y'\hat{j}$$

Displacement vector

$$\Delta\mathbf{r} = \mathbf{r}' - \mathbf{r}$$

$$\Delta\mathbf{r} = (x'\hat{i} + y'\hat{j}) - (x\hat{i} + y\hat{j})$$

$$\Delta\mathbf{r} = (x' - x)\,\hat{i} + (y' - y)\,\hat{j}$$

$$\Delta\mathbf{r} = \Delta x\,\hat{i} + \Delta y\,\hat{j}$$

Velocity vector

$$\mathbf{v} = \frac{\Delta\mathbf{r}}{\Delta t}$$

$$\mathbf{v} = \frac{\Delta x\,\hat{i} + \Delta y\,\hat{j}}{\Delta t}$$

$$\mathbf{v} = \frac{\Delta x}{\Delta t}\hat{i} + \frac{\Delta y}{\Delta t}\hat{j}$$

$$\mathbf{v} = v_x\,\hat{i} + v_y\,\hat{j}$$

Instantaneous velocity

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

$\mathbf{v} = \mathbf{v}_x \hat{i} + \mathbf{v}_y \hat{j}$ where

$$v_x = \frac{dx}{dt}, v_y = \frac{dy}{dt}$$

The magnitude of **v** is then

$$v = \sqrt{v_x^2 + v_y^2}$$

and the direction of **v** is given by the angle θ :

$$\tan\theta = \frac{v_y}{v_x}, \quad \theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

Acceleration

$$\bar{\mathbf{a}} = \frac{\Delta \mathbf{v}}{\Delta t} = \frac{\Delta(v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}})}{\Delta t} = \frac{\Delta v_x}{\Delta t}\hat{\mathbf{i}} + \frac{\Delta v_y}{\Delta t}\hat{\mathbf{j}}$$

$$\mathbf{a} = a_x \hat{i} + a_y \hat{j}$$

Instantaneous Acceleration

$$\mathbf{a} = \frac{d\mathbf{v}}{dt}$$

Or, $\mathbf{a} = a_x \hat{\mathbf{i}} + a_y \hat{\mathbf{j}}$

where, $a_x = \frac{dv_x}{dt}, \; a_y = \frac{dv_y}{dt}$

▶ ***Example 4.4*** The position of a particle is given by

$$\mathbf{r} = 3.0t\,\hat{\mathbf{i}} + 2.0t^2\,\hat{\mathbf{j}} + 5.0\,\hat{\mathbf{k}}$$

where t is in seconds and the coefficients have the proper units for **r** to be in metres. (a) Find **v**(t) and **a**(t) of the particle. (b) Find the magnitude and direction of **v**(t) at $t = 1.0$ s.

Answer

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}\left(3.0\,t\,\hat{\mathbf{i}} + 2.0t^2\,\hat{\mathbf{j}} + 5.0\,\hat{\mathbf{k}}\right)$$

$$= 3.0\hat{\mathbf{i}} + 4.0t\,\hat{\mathbf{j}}$$

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = +4.0\hat{\mathbf{j}}$$

$a = 4.0$ m s^{-2} along y- direction

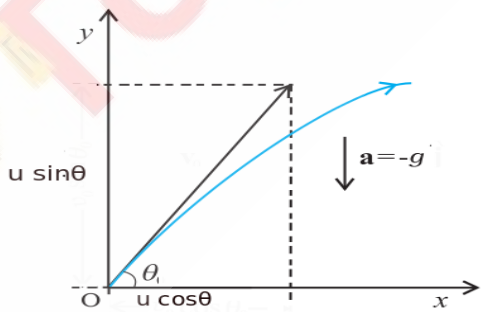
At $t = 1.0$ s, $\mathbf{v} = 3.0\hat{\mathbf{i}} + 4.0\hat{\mathbf{j}}$

It's magnitude is $v = \sqrt{3^2 + 4^2} = 5.0$ m s^{-1}

and direction is

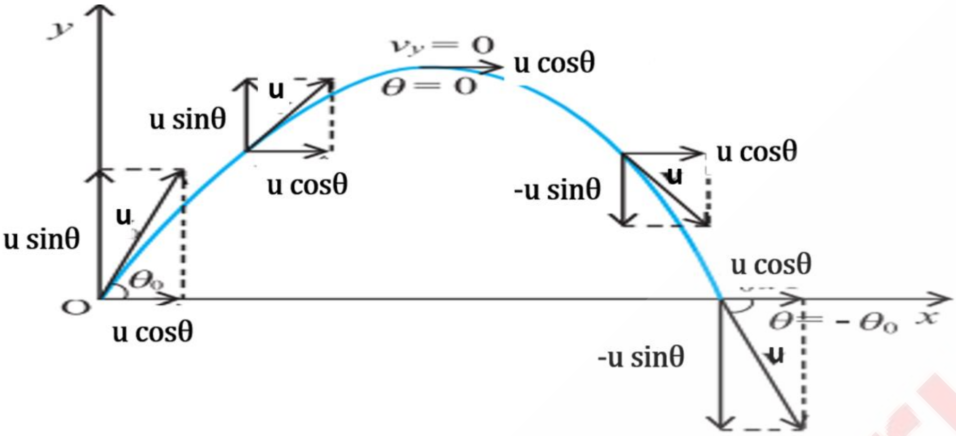
$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{4}{3}\right) \cong 53^{\circ} \text{ with } x\text{-axis.}$$

3.9 Projectile Motion



- An object that is in flight after being thrown or projected is called a projectile.
- The path (trajectory)of a projectile is a parabola.
- The components of initial velocity u are **u cos θ** along horizontal direction and **u sin θ** along vertical direction.

- The x-component of velocity(**u cos θ**) remains constant throughout the motion and hence there is no acceleration in horizontal direction,i.e., $a_x = 0$
- The y- component of velocity (**u sin θ**) changes throughout the motion. At the point of maximum height, **u sin θ** = 0. There is acceleration in vertical direction, $a_y = -g$



Equation of path of a projectile

Displacement of the projectile after a time t

$$x = u\cos\theta\, t$$

$$t = \frac{x}{u\cos\theta}$$

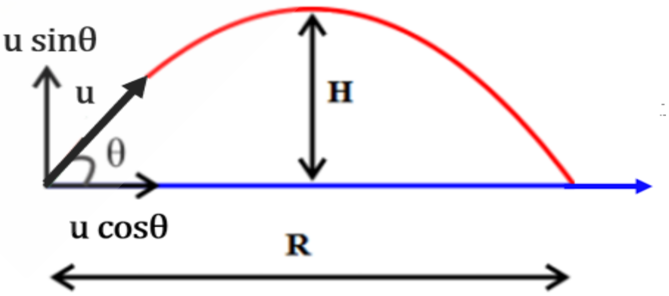
$$y = u\sin\theta\, t - \frac{1}{2} g t^2$$

$$y = u\sin\theta \left(\frac{x}{u\cos\theta}\right) - \frac{1}{2} g \left(\frac{x}{u\cos\theta}\right)^2$$

$$y = \tan\theta\, x - \frac{g}{2u^2\cos^2\theta} x^2$$

This equation is of the form $y = a x + b x^2$, in which a and b are constants. This is the equation of a parabola, i.e. **the path of the projectile is a parabola** .

Time of Flight of a projectile (T)



The total time T during which the projectile is in flight is called Time of Flight, T.

Consider the motion in vertical direction,

$$s = ut + \frac{1}{2}at^2$$

$s = 0$,
$u = u \sin\theta$,
$a = -g$,
$t = T$

$$0 = u\sin\theta\, T - \frac{1}{2}gT^2$$

$$u\sin\theta\, T = \frac{1}{2}gT^2$$

$$T = \frac{2u\sin\theta}{g}$$

Horizontal range of a projectile (R)

The horizontal distance travelled by a projectile during its time of flight is called the horizontal range, R.

Horizontal range = Horizontal component of velocity x Time of flight

$$R = u\cos\theta \times \frac{2u\sin\theta}{g}$$

$$R = \frac{u^2 \times 2\sin\theta\cos\theta}{g}$$

$$R = \frac{u^2\sin 2\theta}{g}$$

R is maximum when $\sin 2\theta$ is maximum, i.e., when $\theta = 45^0$.

$$R_{max} = \frac{u^2}{g}$$

Show that for a given velocity of projection range will be same for angles θ and (90-θ)

For angle θ, $R = \frac{u^2\sin 2\theta}{g}$

For angle $(90-\theta)$, $R = \frac{u^2\sin 2(90-\theta)}{g}$

$$R = \frac{u^2\sin(180-2\theta)}{g}$$

$\sin(180-2\theta) = \sin 2\theta$

$$R = \frac{u^2\sin 2\theta}{g}$$

for given velocity of projection range will be same for angles θ and (90-θ)

Maximum height of a projectile (H)

It is the maximum height reached by the projectile.

Consider the motion in vertical direction to the highest point

$$v^2 - u^2 = 2as$$

$$u = u \sin\theta,$$

$$v = 0,$$

$$a = -g,$$

$$s = H$$

$$0 - u^2 \sin^2\theta = -2gH$$

$$H = \frac{u^2 \sin^2\theta}{2g}$$

Example

A cricket ball is thrown at a speed of 28 m s^{-1} in a direction 30° above the horizontal. Calculate (a) the maximum height, (b) the time taken by the ball to return to the same level, and (c) the distance from the thrower to the point where the ball returns to the same level.

(a) $$H = \frac{u^2 \sin^2\theta}{2g}$$

$$H = \frac{28^2 \sin^2 30}{2 \times 9.8} = 10 \text{ m}$$

(b) $$T = \frac{2u\sin\theta}{g}$$

$$T = \frac{2 \times 9.8 \sin 30}{9.8} = 2.9 \text{ s}$$

(c) $$R = \frac{u^2 \sin 2\theta}{g}$$

$$R = \frac{28^2 \sin 60}{9.8} = 69 \text{ m}$$

3.10 Uniform Circular Motion

When an object follows a circular path at a constant speed, the motion of the object is called **uniform circular motion**. The word "uniform" refers to the speed, which is uniform (constant) throughout the motion.

Period

The time taken by an object to make one revolution is known as its time period T

Frequency

The number of revolutions made in one second is called its frequency.

$$\upsilon = \frac{1}{T}$$

unit - hertz (Hz)

Angular velocity (ω)

angular velocity is the time rate of change of angular displacement

$$\omega = \frac{\Delta\,\theta}{\Delta t}$$

In the limit Δt tends to zero

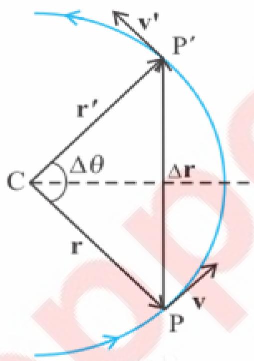
$$\omega = \frac{d\,\theta}{dt}$$

Unit is rad/s

During the time period T ,the angular displacement is 2π radian

$$\omega = \frac{2\pi}{T} \quad \text{or} \quad \omega = 2\pi\,\upsilon$$

Relation connecting angular velocity and linear velocity



As the object moves from P to P′ in time Δt . Δθ is called angular displacement and Δ r is the linear diplacement

$$\text{angle} = \frac{\text{arc}}{\text{radius}}$$

$$\Delta\,\theta = \frac{\Delta\,r}{r}$$

$$\Delta\,r = r\;\Delta\,\theta$$

$$\frac{\Delta\,r}{\Delta t} = r\,\frac{\Delta\,\theta}{\Delta t}$$

$$v = r\,\omega$$

Angular Acceleration

The rate of change of angular velocity is called angular acceleration.

$$\alpha = \frac{d\,\omega}{dt}$$

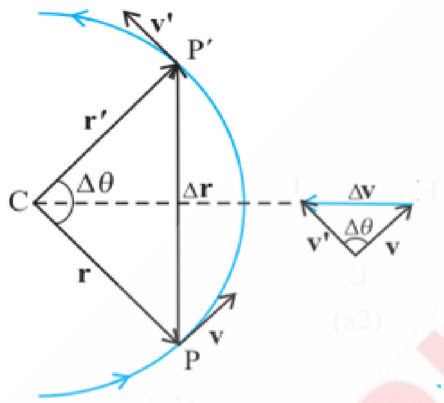
But $\omega = \frac{d\,\theta}{dt}$

$$\alpha = \frac{d}{dt}\left(\frac{d\,\theta}{dt}\right)$$

$$\alpha = \frac{d^2\,\theta}{dt}$$

Centripetal acceleration

A body in uniform circular motion experiences an acceleration , which is directed towards the centre along its radius .This is s called centripetal acceleration .



Let r and r′ be the position vectors and v and v′ the velocities of the object when it is at point P and P ′

$$\frac{\Delta v}{v} = \frac{\Delta r}{r}$$

$$\Delta v = \frac{v\,\Delta r}{r}$$

$$\frac{\Delta v}{\Delta t} = \frac{v\,\Delta r}{r\,\Delta t}$$

$$a = \frac{v}{r} \times v$$

$$a = \frac{v^2}{r}$$

If R is the radius of circular path, then centripetal acceleration .

$$a_c = \frac{v^2}{R}$$

Centripetal acceleration can also be expressed as

$v = R\omega$

$a_c = \frac{v^2}{R}$

$a_c = \frac{R^2\omega^2}{R}$

$a_c = \omega^2 R$

$v = R\omega$

$R = v/\omega$

$a_c = \frac{v^2}{R}$

$a_c = \frac{v^2}{(v/\omega)}$

$a_c = v\omega$

Example

An insect trapped in a circular groove of radius 12 cm moves along the groove steadily and completes 7 revolutions in 100 s.

(a) What is the angular speed, and the linear speed of the motion?
(b) Is the acceleration vector a constant vector ? What is its magnitude ?

Period, $T = \frac{100}{7}$ s

(a) The angular speed ω is given by

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{\frac{100}{7}} = \frac{2\pi \times 7}{100} = 0.44 \text{ rad/s}$$

The linear speed , $v = \omega R = 0.44 \times 0.12 = 5.3 \times 10^{-2}$ m s^{-1}

(b) The direction of velocity v is along the tangent to the circle at every point. The acceleration is directed towards the centre of the circle. Since this direction changes continuously, acceleration here is not a constant vector.

$$a = \omega^2 R = (0.44)^2 \times 0.12 = 2.3 \times 10^{-2} \text{ m s}^{-2}$$