

MISCELLANEOUS EXERCISE

Prove that:

Question 1:

$$2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13} = 0$$

Solution:

$$\begin{aligned} LHS &= 2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13} \\ &= 2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + 2 \cos \left(\frac{\frac{3\pi}{13} + \frac{5\pi}{13}}{2} \right) \cos \left(\frac{\frac{3\pi}{13} - \frac{5\pi}{13}}{2} \right) \left[\because \cos A + \cos B = 2 \cos \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) \right] \\ &= 2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + 2 \cos \frac{4\pi}{13} \cos \left(-\frac{\pi}{13} \right) \\ &= 2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + 2 \cos \frac{4\pi}{13} \cos \frac{\pi}{13} \\ &= 2 \cos \frac{\pi}{13} \left[\cos \frac{9\pi}{13} + \cos \frac{4\pi}{13} \right] \\ &= 2 \cos \frac{\pi}{13} \left[2 \cos \left(\frac{\frac{9\pi}{13} + \frac{4\pi}{13}}{2} \right) \cos \left(\frac{\frac{9\pi}{13} - \frac{4\pi}{13}}{2} \right) \right] \left[\because \cos A + \cos B = 2 \cos \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) \right] \\ &= 2 \cos \frac{\pi}{13} \left[2 \cos \frac{\pi}{2} \cos \frac{5\pi}{26} \right] \\ &= 2 \cos \frac{\pi}{13} \times 2 \times 0 \times \cos \frac{5\pi}{26} \left[\because \cos \frac{\pi}{2} = 0 \right] \\ &= 0 \\ &= RHS \end{aligned}$$

Question 2:

$$(\sin 3x + \sin x) \sin x + (\cos 3x - \cos x) \cos x = 0$$

Solution:

$$\begin{aligned}
LHS &= (\sin 3x + \sin x) \sin x + (\cos 3x - \cos x) \cos x \\
&= \sin 3x \sin x + \sin^2 x + \cos 3x \cos x - \cos^2 x \\
&= \cos 3x \cos x + \sin 3x \sin x - (\cos^2 x - \sin^2 x) \\
&= \cos(3x - x) - \cos 2x & \left[\begin{array}{l} \because \cos(A - B) = \cos A \cos B + \sin A \sin B \\ \& \cos 2A = \cos^2 A - \sin^2 A \end{array} \right] \\
&= \cos 2x - \cos 2x \\
&= 0 \\
&= RHS
\end{aligned}$$

Question 3:

$$(\cos x + \cos y)^2 + (\sin x - \sin y)^2 = 4 \cos^2 \frac{x+y}{2}$$

Solution:

$$\begin{aligned}
LHS &= (\cos x + \cos y)^2 + (\sin x - \sin y)^2 \\
&= \cos^2 x + \cos^2 y + 2 \cos x \cos y + \sin^2 x + \sin^2 y - 2 \sin x \sin y & \left[\begin{array}{l} \because (a+b)^2 = a^2 + b^2 + 2ab \\ \& (a-b)^2 = a^2 + b^2 - 2ab \end{array} \right] \\
&= (\cos^2 x + \sin^2 x) + (\cos^2 y + \sin^2 y) + 2(\cos x \cos y - \sin x \sin y) \\
&= 1 + 1 + 2 \cos(x+y) & \left[\begin{array}{l} \because (\cos^2 A + \sin^2 A) = 1 \\ \& \cos(A+B) = \cos A \cos B - \sin A \sin B \end{array} \right] \\
&= 2 + 2 \cos(x+y) \\
&= 2 \left[1 + \cos 2 \left(\frac{x+y}{2} \right) \right] \\
&= 2 \left[1 + 2 \cos^2 \left(\frac{x+y}{2} \right) - 1 \right] & [\because \cos 2A = 2 \cos^2 A - 1] \\
&= 4 \cos^2 \frac{x+y}{2} \\
&= RHS
\end{aligned}$$

Question 4:

$$(\cos x - \cos y)^2 + (\sin x - \sin y)^2 = 4 \sin^2 \frac{x-y}{2}$$

Solution:

$$\begin{aligned}LHS &= (\cos x - \cos y)^2 + (\sin x - \sin y)^2 \\&= \cos^2 x + \cos^2 y - 2\cos x \cos y + \sin^2 x + \sin^2 y - 2\sin x \sin y \quad \left[\begin{array}{l} \because (a+b)^2 = a^2 + b^2 + 2ab \\ \& (a-b)^2 = a^2 + b^2 - 2ab \end{array} \right] \\&= (\cos^2 x + \sin^2 x) + (\cos^2 y + \sin^2 y) - 2(\cos x \cos y + \sin x \sin y) \\&= 1 + 1 - 2\cos(x-y) \quad \left[\begin{array}{l} \because (\cos^2 A + \sin^2 A) = 1 \\ \& \cos(A-B) = \cos A \cos B + \sin A \sin B \end{array} \right] \\&= 2 - 2\cos(x-y) \\&= 2 \left[1 - \cos 2\left(\frac{x-y}{2}\right) \right] \\&= 2 \left[1 - \left\{ 1 - 2\sin^2\left(\frac{x-y}{2}\right) \right\} \right] \quad \left[\because \cos 2A = 2\cos^2 A - 1 \right] \\&= 2 \left[1 - 1 + 2\sin^2\left(\frac{x-y}{2}\right) \right] \\&= 4\sin^2 \frac{x-y}{2} \\&= RHS\end{aligned}$$

Question 5:

$$\sin x + \sin 3x + \sin 5x + \sin 7x = 4\cos x \cos 2x \sin 4x$$

Solution:

$$\begin{aligned}LHS &= \sin x + \sin 3x + \sin 5x + \sin 7x \\&= (\sin 5x + \sin x) + (\sin 7x + \sin 3x) \\&= \left[2\sin\left(\frac{5x+x}{2}\right)\cos\left(\frac{5x-x}{2}\right) \right] + \left[2\sin\left(\frac{7x+3x}{2}\right)\cos\left(\frac{7x-3x}{2}\right) \right] \\&\quad \left[\because \sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \right] \\&= 2\sin 3x \cos 2x + 2\sin 5x \cos 2x \\&= 2\cos 2x (\sin 5x + \sin 3x) \\&= 2\cos 2x \left[2\sin\left(\frac{5x+3x}{2}\right)\cos\left(\frac{5x-3x}{2}\right) \right] \quad \left[\because \sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \right] \\&= 2\cos 2x [2\sin 4x \cos x] \\&= 4\cos x \cos 2x \sin 4x \\&= RHS\end{aligned}$$

Question 6:

$$\frac{(\sin 7x + \sin 5x) + (\sin 9x + \sin 3x)}{(\cos 7x + \cos 5x) + (\cos 9x + \cos 3x)} = \tan 6x$$

Solution:

$$\begin{aligned} LHS &= \frac{(\sin 7x + \sin 5x) + (\sin 9x + \sin 3x)}{(\cos 7x + \cos 5x) + (\cos 9x + \cos 3x)} \\ &= \frac{2\sin\left(\frac{7x+5x}{2}\right)\cos\left(\frac{7x-5x}{2}\right) + 2\sin\left(\frac{9x+3x}{2}\right)\cos\left(\frac{9x-3x}{2}\right)}{2\cos\left(\frac{7x+5x}{2}\right)\cos\left(\frac{7x-5x}{2}\right) + 2\cos\left(\frac{9x+3x}{2}\right)\cos\left(\frac{9x-3x}{2}\right)} \\ &\quad \left[\begin{array}{l} \because \sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \\ \& \cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \end{array} \right] \\ &= \frac{2\sin 6x \cos x + 2\sin 6x \cos 3x}{2\cos 6x \cos x + 2\cos 6x \cos 3x} \\ &= \frac{2\sin 6x (\cos x + \cos 3x)}{2\cos 6x (\cos x + \cos 3x)} \\ &= \frac{\sin 6x}{\cos 6x} \\ &= \tan 6x \\ &= RHS \end{aligned}$$

Question 7:

$$\sin 3x + \sin 2x - \sin x = 4 \sin x \cos \frac{x}{2} \cos \frac{3x}{2}$$



Solution:

$$LHS = \sin 3x + \sin 2x - \sin x$$

$$= \sin 3x + (\sin 2x - \sin x)$$

$$= \sin 2\left(\frac{3x}{2}\right) + \left[2 \cos\left(\frac{2x+x}{2}\right) \sin\left(\frac{2x-x}{2}\right)\right] \quad \left[\because \sin A - \sin B = 2 \cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)\right]$$

$$= 2 \sin \frac{3x}{2} \cos \frac{3x}{2} + 2 \cos \frac{3x}{2} \sin \frac{x}{2} \quad [\because \sin 2A = 2 \sin A \cos A]$$

$$= 2 \cos \frac{3x}{2} \left(\sin \frac{3x}{2} + \sin \frac{x}{2} \right)$$

$$= 2 \cos \frac{3x}{2} \left[2 \sin \left(\frac{\frac{3x}{2} + \frac{x}{2}}{2} \right) \cos \left(\frac{\frac{3x}{2} - \frac{x}{2}}{2} \right) \right] \quad \left[\because \sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)\right]$$

$$= 2 \cos \frac{3x}{2} \left[2 \sin x \cos \frac{x}{2} \right]$$

$$= 4 \sin x \cos \frac{x}{2} \cos \frac{3x}{2}$$

$$= RHS$$

Find $\sin \frac{x}{2}$, $\cos \frac{x}{2}$ and $\tan \frac{x}{2}$ in each of the following:

Question 8:

$\tan x = -\frac{4}{3}$, x in quadrant II.

Solution:

Since x lies in quadrant II

$$\frac{\pi}{2} < x < \pi$$

Therefore,

$$\frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$$

Hence, $\sin \frac{x}{2}$, $\cos \frac{x}{2}$ and $\tan \frac{x}{2}$ lie in quadrant I and all are positive.

It is given that $\tan x = -\frac{4}{3}$

$$\begin{aligned}\sec^2 x &= 1 + \tan^2 x \\ &= 1 + \left(-\frac{4}{3}\right)^2 \\ &= 1 + \frac{16}{9} \\ &= \frac{25}{9}\end{aligned}$$

$$\sec x = \pm \sqrt{\frac{25}{9}}$$

$$\frac{1}{\cos x} = \pm \frac{5}{3}$$

$$\cos x = \pm \frac{3}{5}$$

As x is in quadrant II, $\cos x$ is negative.

$$\cos x = -\frac{3}{5}$$

$$\cos 2\left(\frac{x}{2}\right) = -\frac{3}{5}$$

$$2\cos^2 \frac{x}{2} - 1 = -\frac{3}{5}$$

$$2\cos^2 \frac{x}{2} = 1 - \frac{3}{5}$$

$$\cos^2 \frac{x}{2} = \frac{2}{5} \times \frac{1}{2}$$

$$\cos^2 \frac{x}{2} = \frac{1}{5}$$

$$\cos \frac{x}{2} = \pm \sqrt{\frac{1}{5}}$$

Since, $\cos \frac{x}{2}$ lies in quadrant I and positive, $\cos \frac{x}{2} = \frac{1}{\sqrt{5}}$

Now,

$$\sin^2 \frac{x}{2} = 1 - \cos^2 \frac{x}{2} \quad \left[\because \sin^2 A + \cos^2 A = 1 \right]$$

$$= 1 - \left(\frac{1}{\sqrt{5}} \right)^2$$

$$= 1 - \frac{1}{5}$$

$$= \frac{4}{5}$$

$$\sin \frac{x}{2} = \pm \sqrt{\frac{4}{5}}$$

$$\sin \frac{x}{2} = \pm \frac{2}{\sqrt{5}}$$

Since, $\sin \frac{x}{2}$ lies in quadrant I and positive, $\sin \frac{x}{2} = \frac{2}{\sqrt{5}}$

Now,

$$\tan \frac{x}{2} = \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}}$$

$$= \frac{\left(\frac{2}{\sqrt{5}} \right)}{\left(\frac{1}{\sqrt{5}} \right)}$$

$$= \frac{2}{\sqrt{5}} \times \frac{\sqrt{5}}{1}$$

$$= 2$$

Therefore, $\sin \frac{x}{2} = \frac{2}{\sqrt{5}}$, $\cos \frac{x}{2} = \frac{1}{\sqrt{5}}$ and $\tan \frac{x}{2} = 2$.

Question 9:

$\cos x = -\frac{1}{3}$, x in quadrant III.

Solution:

Since x lies in quadrant III

$$\pi < x < \frac{3\pi}{2}$$

Therefore,

$$\frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}$$

Hence, $\cos \frac{x}{2}$ and $\tan \frac{x}{2}$ are negative while $\sin \frac{x}{2}$ is positive as all lie in quadrant II.

It is given that $\cos x = -\frac{1}{3}$

$$\cos 2\left(\frac{x}{2}\right) = -\frac{1}{3}$$

$$2\cos^2 \frac{x}{2} - 1 = -\frac{1}{3}$$

$$2\cos^2 \frac{x}{2} = 1 - \frac{1}{3}$$

$$\cos^2 \frac{x}{2} = \frac{2}{3} \times \frac{1}{2}$$

$$\cos^2 \frac{x}{2} = \frac{1}{3}$$

$$\cos \frac{x}{2} = \pm \sqrt{\frac{1}{3}}$$

Since, $\cos \frac{x}{2}$ is negative

So,

$$\begin{aligned} \cos \frac{x}{2} &= -\frac{1}{\sqrt{3}} \\ &= -\frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \\ &= -\frac{\sqrt{3}}{3} \end{aligned}$$

Now,

$$\begin{aligned} \sin^2 \frac{x}{2} &= 1 - \cos^2 \frac{x}{2} \\ &= 1 - \left(-\frac{\sqrt{3}}{3}\right)^2 \\ &= 1 - \frac{1}{3} \\ &= \frac{2}{3} \end{aligned}$$

$$\sin \frac{x}{2} = \pm \sqrt{\frac{2}{3}}$$

Since, $\sin \frac{x}{2}$ positive,

So,

$$[\because \sin^2 A + \cos^2 A = 1]$$



$$\begin{aligned}\sin \frac{x}{2} &= \frac{\sqrt{2}}{\sqrt{3}} \\ &= \frac{\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \\ &= \frac{\sqrt{6}}{3}\end{aligned}$$

Now,

$$\begin{aligned}\tan \frac{x}{2} &= \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} \\ &= \frac{\frac{\sqrt{6}}{3}}{\left(-\frac{\sqrt{3}}{3}\right)} \\ &= \frac{\sqrt{6}}{3} \times \left(-\frac{3}{\sqrt{3}}\right) \\ &= -\sqrt{2}\end{aligned}$$

Therefore, $\sin \frac{x}{2} = \frac{\sqrt{6}}{3}$, $\cos \frac{x}{2} = -\frac{\sqrt{3}}{3}$ and $\tan \frac{x}{2} = -\sqrt{2}$.

Question 10:

$\sin x = \frac{1}{4}$, x in quadrant II.

Solution:

Since x lies in quadrant II

$$\frac{\pi}{2} < x < \pi$$

Therefore,

$$\frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$$

Hence, $\sin \frac{x}{2}$, $\cos \frac{x}{2}$ and $\tan \frac{x}{2}$ lie in quadrant I and all are positive.

It is given that $\sin x = \frac{1}{4}$

Therefore,

$$\begin{aligned}
 \cos^2 x &= 1 - \sin^2 x & [\because \sin^2 A + \cos^2 A = 1] \\
 &= 1 - \left(\frac{1}{4}\right)^2 \\
 &= 1 - \frac{1}{16} \\
 &= \frac{15}{16} \\
 \cos x &= \pm \sqrt{\frac{15}{16}}
 \end{aligned}$$

Since, $\cos x$ lies in quadrant II and negative
So,

$$\begin{aligned}
 \cos x &= -\sqrt{\frac{15}{16}} \\
 \cos 2\left(\frac{x}{2}\right) &= -\frac{\sqrt{15}}{4} \\
 2\cos^2 \frac{x}{2} - 1 &= -\frac{\sqrt{15}}{4} \\
 2\cos^2 \frac{x}{2} &= 1 - \frac{\sqrt{15}}{4} \\
 2\cos^2 \frac{x}{2} &= \frac{4 - \sqrt{15}}{4} \\
 \cos^2 \frac{x}{2} &= \frac{4 - \sqrt{15}}{8} \\
 \cos \frac{x}{2} &= \pm \sqrt{\frac{4 - \sqrt{15}}{8}}
 \end{aligned}$$

Since, $\cos \frac{x}{2}$ lies in quadrant I and is positive.

So,

$$\begin{aligned}
 \cos \frac{x}{2} &= \sqrt{\frac{4 - \sqrt{15}}{8}} \\
 &= \sqrt{\frac{4 - \sqrt{15}}{8}} \times \frac{2}{2} \\
 &= \frac{\sqrt{8 - 2\sqrt{15}}}{4}
 \end{aligned}$$

Now,

$$\sin^2 \frac{x}{2} = 1 - \cos^2 \frac{x}{2} \quad \left[\because \sin^2 A + \cos^2 A = 1 \right]$$

$$= 1 - \left(\frac{\sqrt{8-2\sqrt{15}}}{4} \right)^2$$

$$= 1 - \frac{8-2\sqrt{15}}{16}$$

$$= \frac{16-8+2\sqrt{15}}{16}$$

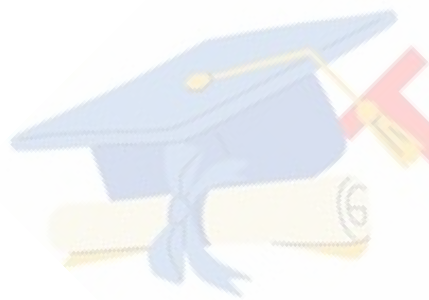
$$= \frac{8+2\sqrt{15}}{16}$$

$$\sin \frac{x}{2} = \pm \sqrt{\frac{8+2\sqrt{15}}{16}}$$

Since, $\sin \frac{x}{2}$ positive,

So,

$$\sin \frac{x}{2} = \frac{\sqrt{8+2\sqrt{15}}}{4}$$



Now,

$$\begin{aligned}\tan \frac{x}{2} &= \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} \\&= \frac{\left(\frac{\sqrt{8+2\sqrt{15}}}{4} \right)}{\left(\frac{\sqrt{8-2\sqrt{15}}}{4} \right)} \\&= \left(\frac{\sqrt{8+2\sqrt{15}}}{4} \right) \times \left(\frac{4}{\sqrt{8-2\sqrt{15}}} \right) \\&= \sqrt{\frac{2(4+\sqrt{15})}{2(4-\sqrt{15})}} \\&= \sqrt{\frac{4+\sqrt{15}}{4-\sqrt{15}} \times \frac{4+\sqrt{15}}{4+\sqrt{15}}} \\&= \sqrt{\frac{(4+\sqrt{15})^2}{16-15}} \\&= 4+\sqrt{15}\end{aligned}$$

Therefore, $\sin \frac{x}{2} = \frac{\sqrt{8+2\sqrt{15}}}{4}$, $\cos \frac{x}{2} = \frac{\sqrt{8-2\sqrt{15}}}{4}$ and $\tan \frac{x}{2} = 4+\sqrt{15}$.

