Chapter 13
Oscillations

Non Periodic Motion

The motion which is non-repetitive .
e.g. rectilinear motion , motion of a projectile.

Periodic Motion

A motion that repeats itself at regular intervals of time is called periodic motion.
e.g. uniform circular motion , orbital motion of planets in the solar system.

Oscillatory Motion

Periodic to and fro motion is called oscillatory motion.
e.g. motion of a cradle , motion of a swing, motion of the pendulum of a wall clock.

Every oscillatory motion is periodic, but every periodic motion need not be oscillatory.

Oscillations and Vibration

There is no significant difference between oscillations and vibrations.

- **When the frequency is small, we call it oscillation.**

  e.g.The oscillation of a branch of a tree
- **When the frequency is high, we call it vibration.**

  e.g. The vibration of a string of a musical instrument.

Period and frequency

Period (T)

The period T is the time required for one complete oscillation, or cycle.

Its SI unit is second.

Frequency

The frequency ν of periodic or oscillatory motion is the number of oscillations per unit time.

It is the reciprocal of period .

$$\nu = \frac{1}{T}$$

The SI unit of ν is hertz (Hz).
(In honor of the discoverer of radio waves, Heinrich Rudolph Hertz)

$$1\text{Hz} = 1\text{oscillation per second} = 1\text{s}^{-1}$$

Example

On an average a human heart is found to beat 75 times in a minute. Calculate its frequency and period.

$$\text{The beat frequency of heart}, \nu = \frac{75}{1\text{min}}$$

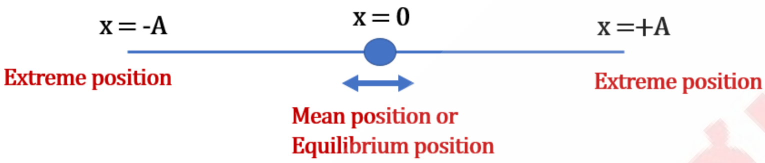
$$= \frac{75}{60\text{ s}}$$

$$= 1.25\text{ s}^{-1} = 1.25\text{ Hz}$$

$$\text{The time period}, T = \frac{1}{1.25}$$

$$T = 0.8\text{ s}$$

Displacement



The distance from mean position is called displacement (x)

At mean position displacement x= 0 and at extreme position $x = \pm A$

A is called amplitude of oscillation.

Amplitude

The maximum displacement from the mean poition is called amplitude (A) of oscillation.

Mathematical Expression for Displacement

The displacement can be represented by a mathematical function of time. It can be a sine function, cosine function or a linear combination of sine and cosine functions.

$$f(t) = A\cos\omega t \quad \text{or}$$

$$f(t) = A\sin\omega t.$$

$$f(t) = A\sin\omega t + B\cos\omega t$$

Where A = Amplitude

ω=angular frequency

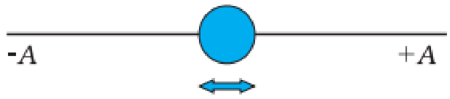
$$\omega = \frac{2\pi}{T} \text{ or } \omega = 2\pi\nu$$

Simple Harmonic Motion

Simple harmonic motion is the simplest form of oscillatory motion.

A particle is said to be in simple harmonic motion ,if the force acting on the particle is proportional to its displacement and is directed towards the mean position.

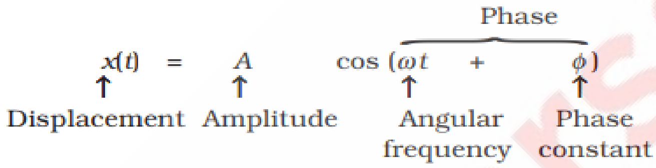
Mathematical expression for an SHM



Consider a particle vibrating back and forth about the origin of x-axis, between the limits +A and –A.

If the motion is simple harmonic ,its position can be represented as a function of time.

$$x(t) = A\cos(\omega t + \phi)$$



Phase

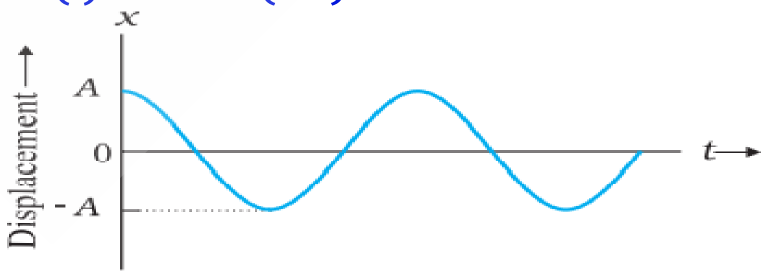
The time varying quantity, **$(\omega t + \phi)$**, is called the phase of the motion.
It describes the state of motion at a given time.

Phase Constant

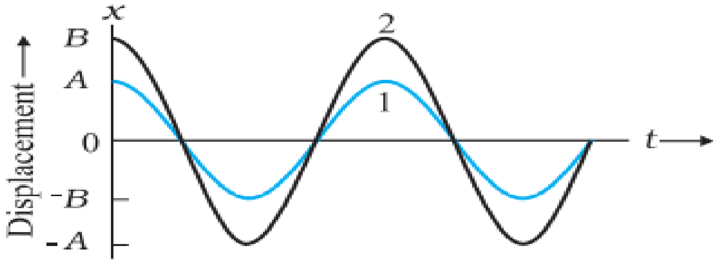
The constant ϕ is called the phase constant (or phase angle). The value of ϕ depends on the displacement and velocity of the particle at $t = 0$.
The phase constant signifies the initial conditions.

A plot of displacement as a function of time for $\phi = 0$.

$$x(t) = A\cos(\omega t)$$



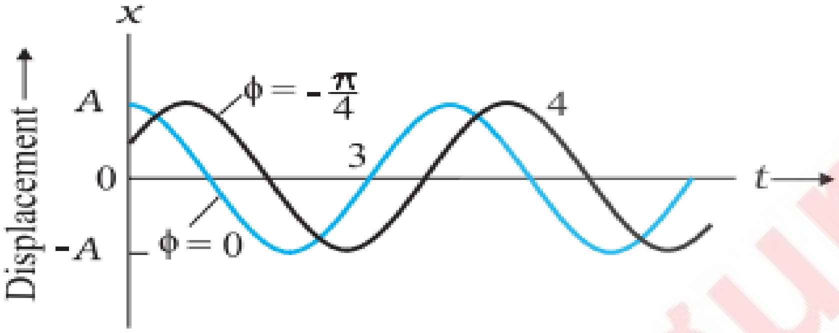
The curves 1 and 2 are for two different amplitudes A and B.



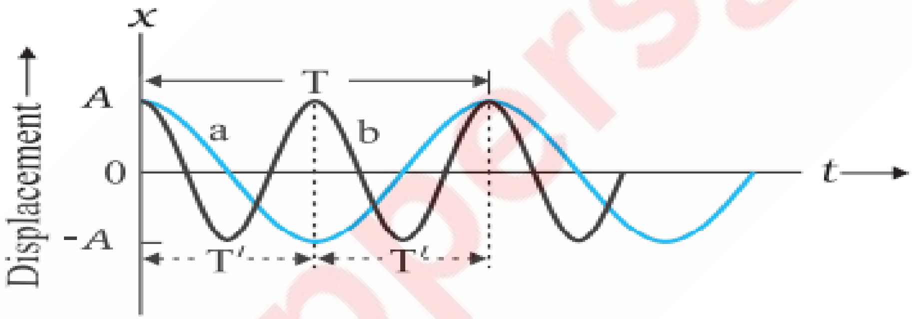
The curve 3 , for $\phi = 0$, $x(t) = A\cos(\omega t)$

The curve 4 , for $\phi = -\pi/4$, $x(t) = A\cos(\omega t - \pi/4)$

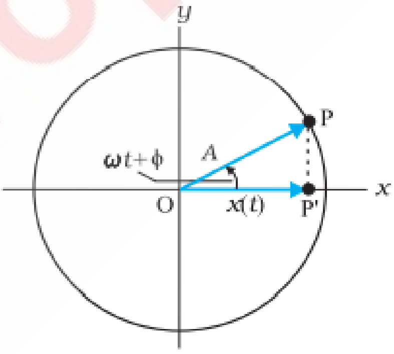
The amplitude A is same for both the plots



Plots of for $\phi = 0$ for two different periods.



Simple Harmonic Motion and Uniform Circular Motion



Consider a particle P in uniform circular motion.

The projection of particle along a diameter of the circle is x(t) .

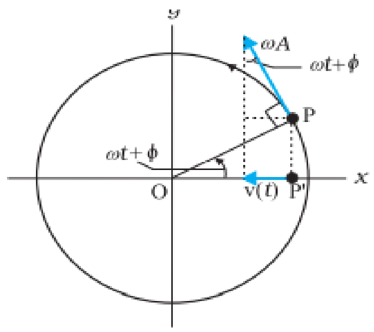
From figure, $\cos(\omega t + \phi) = \frac{x(t)}{A}$

$$x(t) = A\cos(\omega t + \phi) \text{ ----------------(1)}$$

This equation represents a Simple Harmonic Motion.

i. e, the projection of uniform circular motion on a diameter of the circle is in Simple Harmonic Motion.

Velocity in Simple Harmonic Motion



The speed of a particle v in uniform circular motion is its angular speed ω times the radius of the circle A.

$$v = \omega A$$

Its projection on the x-axis gives the velocity in SHM

$$v(t) = -\omega A \sin(\omega t + \phi)$$

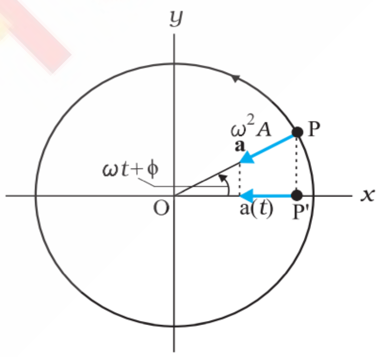
The negative sign appears because the velocity component of P is directed towards the left, in the negative direction of x.

This equation expresses the instantaneous velocity of the particle P′ (projection of P). Therefore, it expresses the instantaneous velocity of a particle executing SHM.

The velocity in SHM can also be obtained by differentiating x(t) with respect to time .

$$v(t) = \frac{d}{dt} x(t)$$

Acceleration in SHM



A particle executing a uniform circular motion is subjected to a radial acceleration ,which is directed towards the centre.

$$a = \omega^2 A$$

Its projection on the x-axis gives the acceleration of SHM.

$$a(t) = -\omega^2 A \cos(\omega t + \phi)$$

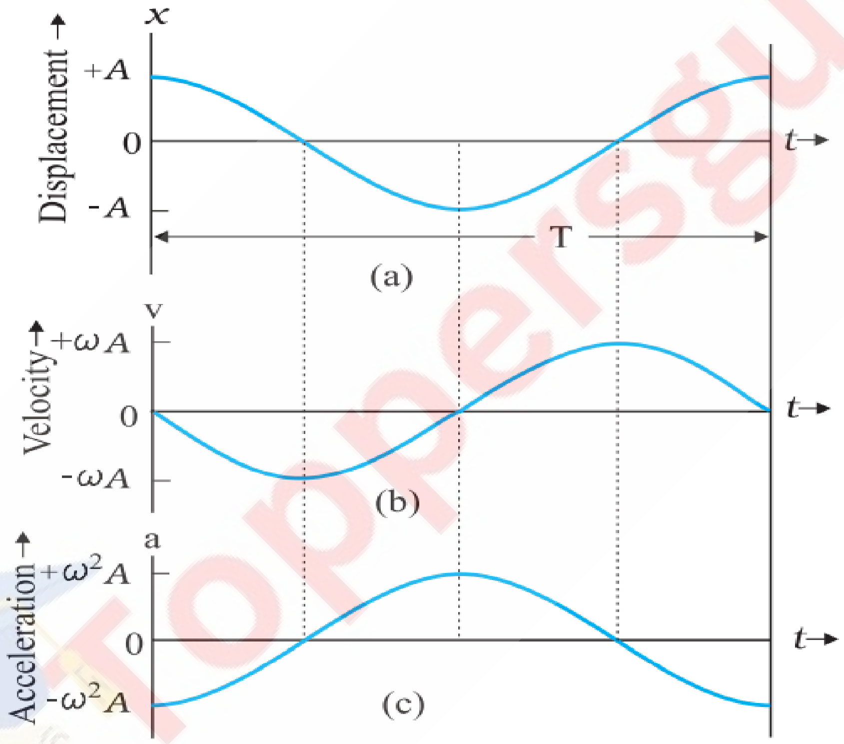
$$a(t) = -\omega^2 x \text{ ------------(3)}$$

The acceleration, a(t), of the particle P′ is the projection of the acceleration a of the reference particle P. It is an important result for SHM. **It shows that in SHM, the acceleration is proportional to the displacement and is always directed towards the mean position.**

The acceleration in SHM can also be obtained by differentiating v(t) with respect to time

$$a(t) = \frac{d}{dt} v(t)$$

The variation of particle displacement, velocity and acceleration in a simple harmonic motion



- The particle velocity lags behind the displacement by a phase angle of π/2; when the magnitude of displacement is the greatest, the magnitude of the velocity is the least. When the magnitude of displacement is the least, the velocity is the greatest.
- When the displacement has its greatest positive value, the acceleration has its greatest negative value and vice versa. When the displacement is zero, the acceleration is also zero.

Force Law for Simple Harmonic Motion

$$F = ma$$

$$a = -\omega^2 x$$

$$F = -m\omega^2 x$$

$$F = -kx \text{ -------------(4)}$$

Where $k = m\omega^2$; $\omega^2 = \frac{k}{m}$

$$\omega = \sqrt{\frac{k}{m}}$$

The force in SHM is proportional to the displacement and its direction is opposite to the direction of displacement. Therefore, it is a restoring force.

Note:

- The centripetal force for uniform circular motion is constant in magnitude, but the restoring force for SHM is time dependent.
- Since the force F is proportional to x such a system is also referred to as a linear harmonic oscillator.

Energy in Simple Harmonic Motion

A particle executing simple harmonic motion has kinetic and potential energies, both varying between the limits, zero and maximum.

Kinetic Energy in Simple Harmonic Motion

$$K = \frac{1}{2}mv^2$$

$$K = \frac{1}{2}mv^2$$

$$v = -\omega\sqrt{A^2 - x^2}$$

$$v^2 = \omega^2(A^2 - x^2)$$

$$K = \frac{1}{2}m\omega^2(A^2 - x^2) \text{ ---------------(5)}$$

- At mean position $(x=0)$, $K = \frac{1}{2}m\omega^2 A^2$

  KE is maximum At Mean position
- At extreme position $(x=A)$, $K = 0.$

  KE is minimum At extreme positions.

Thus the kinetic energy of a particle executing simple harmonic motion is periodic, with period T/2.

Potential Energy in Simple Harmonic Motion

$$U = \frac{1}{2}kx^2$$

$$k = m\,\omega^2$$

$$\mathbf{U = \frac{1}{2}m\omega^2x^2} \text{ ---------------(6)}$$

- At Mean position (x=0) ,**U =0**

PE is minimum At Mean position

- At Extreme position (x=A) , $\mathbf{U = \frac{1}{2}m\omega^2A^2}$

PE is maximum At extreme positions.

Thus the potential energy of a particle executing simple harmonic motion is also periodic, with period T/2.

The Total Energy in SHM

$$E = U + K$$

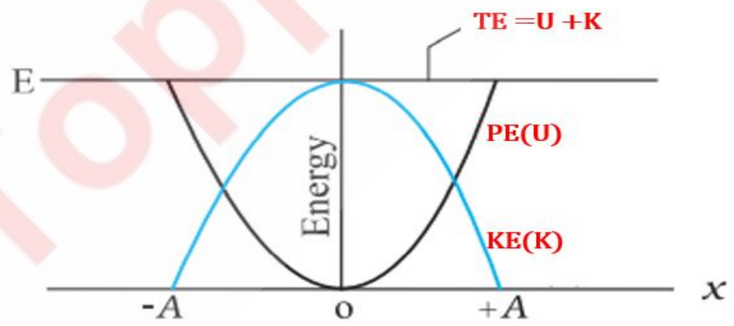
$$E = \frac{1}{2}m\omega^2x^2 + \frac{1}{2}m\omega^2(A^2 - x^2)$$

$$E = \frac{1}{2}m\omega^2x^2 + \frac{1}{2}m\omega^2A^2 - \frac{1}{2}m\omega^2x^2$$

$$\mathbf{E = \frac{1}{2}m\omega^2A^2} \text{ --------------(7)}$$

The total mechanical energy of a harmonic oscillator is a constant or independent of time.

Variation of Potential energy , kinetic energy K and the total energy E with time t for a linear harmonic oscillator



At what position the KE of a simple harmonic oscillator becomes **equal to its potential energy?**

$$KE = PE$$

$$\frac{1}{2}m\omega^2(A^2 - x^2) = \frac{1}{2}m\omega^2x^2$$

$$A^2 - x^2 = x^2$$

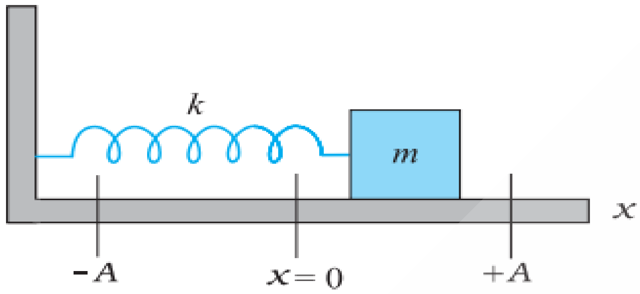
$$A^2 = 2x^2$$

$$x^2 = \frac{A^2}{2}, \quad \mathbf{x = \frac{A}{\sqrt{2}}}$$

Some Systems Executing Simple Harmonic Motion

There are no physical examples of absolutely pure simple harmonic motion. In practice we come across systems that execute simple harmonic motion approximately under certain conditions.

Oscillations due to a Spring



The small oscillations of a block of mass m fixed to a spring, is fixed to a rigid wall is an example of SHM.

The restoring force F acting on the block is, $F(x) = -k x$

k, is called the spring constant.

A stiff spring has large k and a soft spring has small k.

Equation is same as the eqn for force in SHM and therefore the spring executes a simple harmonic motion.

Period of Oscillations of a Spring

Restoring force, $F = -k x$

Where $k = m\omega^2$

$$\omega^2 = \frac{k}{m}$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$\text{Period, } T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{k}{m}}}$$

$$\mathbf{T = 2\pi\sqrt{\frac{m}{k}}}$$

Example

A 5 kg collar is attached to a spring of spring constant 500 N m^{-1}. It slides without friction over a horizontal rod. The collar is displaced from its equilibrium position by 10.0 cm and released. Calculate
(a) the period of oscillation,
(b) the maximum speed and
(c) maximum acceleration of the collar.

(a) The period of oscillation as given by

$$T = 2\pi\sqrt{\frac{m}{k}}$$

$$T = 2\pi\sqrt{\frac{5}{500}}$$

$$T = 2x\ 3.14\ x\frac{1}{10}$$

$$= 0.63\ s$$

(b) The velocity of the collar executing SHM is

$$v = -\ \omega\sqrt{A^2 - x^2}$$

Maximum speed, $v = A\omega$ (at mean position ,x=0)

$$\omega = \sqrt{\frac{k}{m}}$$

$$v = A\sqrt{\frac{k}{m}}$$

$$A = 10cm = 0.1m$$

$$v = 0.1\ x\sqrt{\frac{500}{5}}$$

$$v = 0.1\ x10 = 1\ m/s$$

(c) Acceleration in SHM

$$a = -\ \omega^2 x$$

Maximum acceleration, $a = \omega^2 A$ (at extreme position)

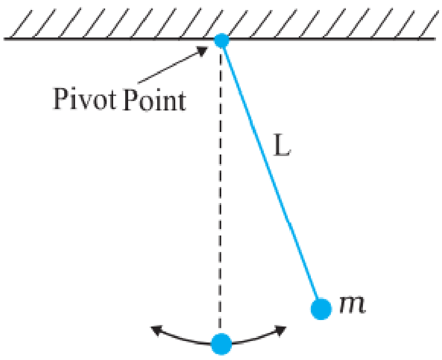
$$\omega^2 = \frac{k}{m}$$

$$a = \frac{k}{m}A$$

$$a = \frac{500}{5}x\ 0.1$$

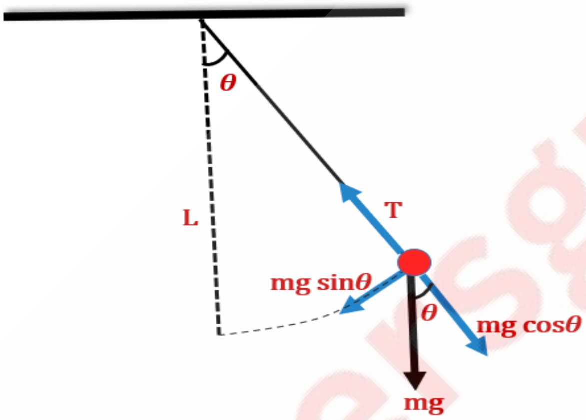
$$a = 10m/s^2$$

The Simple Pendulum



A simple pendulum consists of a particle of mass m (bob) suspended from one end of an unstretchable, massless string of length L fixed at the other end.

Period of Oscillations of a Simple Pendulum



The radial component, **mg cosθ** is cancelled by the tension,T.
The tangential component, mg sinθ produces a restoring torque,

$$\tau = -L\,(mg \sin\theta\,) \text{ --------------(1)}$$

(Where the negative sign indicates that the torque acts to reduce θ.)

L= length of simple pendulum.

For rotational motion we have,

$$\tau = I\,\alpha \text{ -----------------(2)}$$

α is angular acceleration.

From eqn (1) and (2)

$$I\,\alpha = -L\,mg \sin\theta$$

$$\alpha = \frac{-mgL}{I} \sin\theta \qquad \text{(since } \theta \text{ is very small, } \sin\theta \approx \theta)$$

$$\alpha = \frac{-mgL}{I}\,\theta \text{ ---------------(3)}$$

Acceleration of SHM , $a = -\,\omega^2 x$ ------------------(4)

Comparing eqns (3) and (4)

$$\omega^2 = \frac{mgL}{I}$$

$$I = mL^2$$

$$\omega^2 = \frac{mgL}{mL^2}$$

$$\omega^2 = \frac{g}{L}$$

$$\omega = \sqrt{\frac{g}{L}}$$

Period, $T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{g}{L}}}$

$$\mathbf{T = 2\pi\sqrt{\frac{L}{g}}}$$

Example

What is the length of a simple pendulum, which ticks seconds (seconds pendulum) ?

$$T = 2\pi\sqrt{\frac{L}{g}}$$

$$T^2 = 4\pi^2\frac{l}{g}$$

$$L = \frac{T^2 g}{4\pi^2}$$

For seconds pendulum ,$T = 2s$

$$L = \frac{2^2 \text{ x } 9.8}{4 \text{ x } 3.14^2} = 0.994 \approx 1m$$