MISCELLANEOUS EXERCISE

Question 1:

Three vertices of a parallelogram ABCD are $A(3,-1,2), B(1,2,-4)$ and $C(-1,1,2)$. Find the coordinates of the fourth vertex.

Solution:

The three vertices of a parallelogram ABCD are given as $A(3,-1,2), B(1,2,-4)$ and $C(-1,1,2)$.
Let the coordinates of the fourth vertex be $D(x,y,z)$.

We know that the diagonals of a parallelogram bisect each other.
Therefore, in parallelogram ABCD, diagonals AC and BD bisect each other.

i.e., Mid-point of AC = Mid-point of BD

$$\Rightarrow \left(\frac{3-1}{2},\frac{-1+1}{2},\frac{2+2}{2}\right)=\left(\frac{x+1}{2},\frac{y+2}{2},\frac{z-4}{2}\right)$$

$$\Rightarrow (1,0,2)=\left(\frac{x+1}{2},\frac{y+2}{2},\frac{z-4}{2}\right)$$

Hence,

$$\frac{x+1}{2}=1, \frac{y+2}{2}=0 \text{ and } \frac{z-4}{2}=2$$

$$\Rightarrow x=1, y=-2 \text{ and } z=8$$

Thus, the coordinates of the fourth vertex D are $(1,-2,8)$.

Question 2:

Find the lengths of the medians of the triangle with vertices $A(0,0,6), B(0,4,0)$ and $C(6,0,0)$.

Solution:

Let AD, BE and CF be the medians of the given triangle.

Since, AD is the median, D is the mid-point of BC

Coordinates of point $D=\left(\frac{0+6}{2},\frac{4+0}{2},\frac{0+0}{2}\right)=(3,2,0)$

$$
\begin{aligned}
AD &= \sqrt{(0-3)^2+(0-2)^2+(6-0)^2} \\
&= \sqrt{9+4+36} \\
&= \sqrt{49} \\
&= 7
\end{aligned}
$$

Since, BE is the median, E is the mid-point of AC

Coordinates of point $E=\left(\frac{0+6}{2},\frac{0+0}{2},\frac{0+6}{2}\right)=(3,0,3)$

$$
\begin{aligned}
BE &= \sqrt{(3-0)^2+(0-4)^2+(3-0)^2} \\
&= \sqrt{9+16+9} \\
&= \sqrt{34}
\end{aligned}
$$

Since CF is the median, F is the mid-point of AB

Coordinates of point $F=\left(\frac{0+0}{2},\frac{0+4}{2},\frac{0+6}{2}\right)=(0,2,3)$

$$
\begin{aligned}
CF &= \sqrt{(6-0)^2+(0-2)^2+(0-3)^2} \\
&= \sqrt{36+4+9} \\
&= \sqrt{49} \\
&= 7
\end{aligned}
$$

Thus, the lengths of the medians of triangle ABC are $7, \sqrt{34}$ and 7.

Question 3:

If the origin is the centroid of the triangle PQR with vertices $P(2a,2,6), Q(-4,3b,-10)$ and $R(8,14,2c)$, then find the values of a, b and c.

Solution:

It is known that the coordinates of the centroid of the triangle, whose vertices are $(x_1,y_1,z_1),(x_2,y_2,z_2)$ and (x_3,y_3,z_3) are $\left(\frac{x_1+x_2+x_3}{2},\frac{y_1+y_2+y_3}{2},\frac{z_1+z_2+z_3}{2}\right)$

Therefore, coordinates of the centroid of

$$
\begin{aligned}
\Delta PQR &= \left(\frac{2a-4+8}{3},\frac{2+3b+14}{3},\frac{6-10+2c}{3}\right) \\
&= \left(\frac{2a+4}{3},\frac{3b+16}{3},\frac{2c-4}{3}\right)
\end{aligned}
$$

It is given that origin $(0,0,0)$ is the centroid of the ΔPQR

Hence,

$$\frac{2a+4}{3}=0, \frac{3b+16}{3} \text{ and } \frac{2c-4}{3}=0$$

Thus, the values of $a=-2, b=-\frac{16}{3}$ and $c=2$.

Question 4:

Find the coordinates of a point on $y-$axis which are at distance of $5\sqrt{2}$ from the point $P(3,-2,5)$.

Solution:

If a point is on the $y-$axis, then $x-$coordinate and the $z-$coordinate of the point are zero.

Let $A(0,b,0)$ be the point on the $y-$axis at a distance of $5\sqrt{2}$ from point $P(3,-2,5)$.

Accordingly, $AP=5\sqrt{2}$

On squaring both sides, we obtain $AP^2=50$

Therefore,

$$(3-0)^2+(-2-b)^2+(5-0)^2=50$$
$$9+4+b^2+4b+25=50$$
$$b^2+4b-12=0$$
$$b^2+6b-2b-12=0$$
$$(b+6)(b-2)=0$$

$$\Rightarrow b=-6 \text{ and } b=2$$

Thus, the coordinates of the required point are $(0,2,0)$ and $(0,-6,0)$.

Question 5:

A point R with $x-$coordinate 4 lies on the line segment joining the points $P(2,-3,4)$ and $Q(8,0,10)$. Find the coordinates of the point R.

[Hint Suppose R divides PQ in the ratio $k:1$. The coordinates of the point R are given by $\left(\frac{8k+2}{k+1}, \frac{-3}{k+1}, \frac{10k+4}{k+1}\right)$]

Solution:

The coordinates of points P and Q are given as $P(2,-3,4)$ and $Q(8,0,10)$.
Let R divide the line segment PQ in the ratio $k:1$.

Hence, by section formula, the coordinates of point R are given by

$$\left(\frac{k(8)+2}{k+1},\frac{k(0)-3}{k+1},\frac{k(10)+4}{k+1}\right)$$
$$\Rightarrow\left(\frac{8k+2}{k+1},\frac{-3}{k+1},\frac{10k+4}{k+1}\right)$$

It is given that the $x-$coordinate of point R is 4.

Hence,

$$\frac{8k+2}{k+1}=4$$
$$8k+2=4k+4$$
$$4k=2$$
$$k=\frac{1}{2}$$

Therefore, the coordinates of point R are

$$\left(4,\frac{-3}{\frac{1}{2}+1},\frac{10\left(\frac{1}{2}\right)+4}{\frac{1}{2}+1}\right)$$
$$\Rightarrow(4,-2,6)$$

Thus, the coordinates of point R are $(4,-2,6)$.

Question 6:

If A and B be the points $(3,4,5)$ and $(-1,3,-7)$ respectively, find the equation of the set of points P such that $PA^2+PB^2=k^2$, where k is a constant.

Solution:

The coordinates of points A and B are given as $A(3,4,5)$ and $B(-1,3,-7)$ respectively.

Let the coordinates of point P be (x,y,z).
On using distance formula, we obtain

$$\begin{aligned} PA^2 &= (x-3)^2 + (y-4)^2 + (z-5)^2 \\ &= x^2 + 9 - 6x + y^2 + 16 - 8y + z^2 + 25 - 10z \\ &= x^2 - 6x + y^2 - 8y + z^2 - 10z + 50 \end{aligned}$$

$$\begin{aligned} PB^2 &= (x+1)^2 + (y-3)^2 + (z+7)^2 \\ &= x^2 + 1 + 2x + y^2 + 9 - 6y + z^2 + 49 + 14z \\ &= x^2 + 2x + y^2 - 6y + z^2 + 14z + 59 \end{aligned}$$

Now, if $PA^2 + PB^2 = k^2$, then,

$$\Rightarrow \left(x^2 - 6x + y^2 - 8y + z^2 - 10z + 50\right) + \left(x^2 + 2x + y^2 - 6y + z^2 + 14z + 59\right) = k^2$$

$$\Rightarrow 2x^2 + 2y^2 + 2z^2 - 4x - 14y + 4z + 109 = k^2$$

$$\Rightarrow 2\left(x^2 + y^2 + z^2 - 2x - 7y + 2z\right) = k^2 - 109$$

$$\Rightarrow x^2 + y^2 + z^2 - 2x - 7y + 2z = \frac{k^2 - 109}{2}$$

Thus, the required equation is $x^2 + y^2 + z^2 - 2x - 7y + 2z = \frac{k^2 - 109}{2}$.