Expand each of the expressions in Exercises 1 to 5.

Q1. $(1-2x)^5$

A.1. $(1-2x)^5$

By using binomial theorem we have,

$$= {}^5C_0 (1)^5 + {}^5C_1 (1)^4 (-2x) + {}^5C_2 (1)^3 (-2x)^2 + {}^5C_3 (1)^2 (-2x)^3 + {}^5C_4 (1)^1 (-2x)^4 + {}^5C_5 (-2x)^5$$

$$= \left(\frac{5!}{0!\,(5-0)!}\times 1\right) + \left[\frac{5!}{1!\,(5-1)!}\times 1\times(-2x)\right] + [x1\times(4x^2)] + \left[\frac{5!}{3!\,(5-3)!}\times 1\times(-8x^3)\right] +$$

$$\left[\frac{5!}{4!\,(5-4)!}\times 1\times(16x^4)\right] + \left[\frac{5!}{5!\,(5-5)!}\times(-32x^5)\right]$$

$$= 1 + \left[\frac{5\times 4!}{1\times 4!}\times(-2x)\right] + \left[\frac{5\times 4\times 3!}{2\times 1\times 3!}\times 4x^2\right] + \left[\frac{5\times 4\times 3!}{3!\times 2!}\times(-8x^3)\right] + \left[\frac{5\times 4!}{4!\times 1}\times 16x^4\right] + [1\times(-32x^5)]$$

$$= 1 + [5\times(-2x)] + [10\times 4x^2] + [10\times(-8x^3)] + [5\times 16x^4] + [-32x^5]$$

$$= 1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$$

Q2. $\left(\frac{2}{x}-\frac{x}{2}\right)^5$

A.2. $\left(\frac{2}{x}-\frac{x}{2}\right)^5$

Using $(x-y)^n$

$$= {}^5C_0\left(\frac{2}{x}\right)^5 - {}^5C_1\left(\frac{2}{x}\right)^4\left(\frac{x}{2}\right) + {}^5C_2\left(\frac{2}{x}\right)^3\left(\frac{x}{2}\right)^2 - {}^5C_3\left(\frac{2}{x}\right)^2\left(\frac{x}{2}\right)^3 + {}^5C_4\left(\frac{2}{x}\right)\left(\frac{x}{2}\right)^4 - {}^5C_5\left(\frac{x}{2}\right)^5$$

$$= \left[\frac{5!}{0!\,(5-0)!}\times\frac{32}{x^5}\right] - \left[\frac{5!}{1!\,(5-1)!}\times\frac{16}{x^4}\times\frac{x}{2}\right] + \left[\frac{5!}{2!\,(5-2)!}\times\frac{8}{x^3}\times\frac{x^2}{4}\right] - \left[\frac{5!}{3!\,(5-3)!}\times\frac{4}{x^2}\times\frac{x^3}{8}\right] +$$

$$\left[\frac{5!}{4!\,(5-4)!}\times\frac{x}{2}\times\frac{x^4}{16}\right] - \left[\frac{5!}{5!\,(5-5)!}\times\frac{x^5}{32}\right]$$

$$= \left[1\times\frac{32}{x^5}\right] - \left[\frac{5\times 4!}{1\times 4!}\times\frac{8}{x^3}\right] + \left[\frac{5\times 4\times 3!}{2\times 1\times 3!}\times\frac{2}{x}\right] - \left[\frac{5\times 4\times 3!}{3!\times 2!}\times\frac{x}{2}\right] + \left[\frac{5\times 4!}{4!\times 1!}\times\frac{x^3}{8}\right] - \left[1\times\frac{x^5}{32}\right]$$

$$= \frac{32}{x^5} - \frac{40}{x^3} + \frac{20}{x} - 5x + \frac{5}{8}x^3 - \frac{x^5}{32}$$

Q3. $(2x-3)^6$

A.3. $(2x-3)^6$

By using binomial theorem,

$(2x-3)^6$

$= {}^6C_0\,(2x)^6 + {}^6C_1\,(2x)^5(-3) + {}^6C_2\,(2x)^4(-3)^2 + {}^6C_3\,(2x)^3(-3)^3 + {}^6C_4\,(2x)^2(-3)^4 + {}^6C_5\,(2x)(-3)^5 + {}^6C_6\,(-3)^6$

$$= \left[\frac{6!}{0!\,(6-0)!}\times 64x^6\right] + \left[\frac{6!}{1!\,(6-1)!}\times 32x^5\times(-3)\right] + \left[\frac{6!}{2!\,(6-2)!}\times 16x^4\times 9\right] +$$

$$\left[\frac{6!}{3!\,(6-3)!}\times 8x^3\times(-27)\right] + \left[\frac{6!}{4!\,(6-4)!}\times 4x^2\times 81\right] + \left[\frac{6!}{5!\,(6-5)!}\times 2x\times(-243)\right] + \left[\frac{6!}{6!\,(6-6)!}\times 729\right]$$

$$= [1\times 64x^6] + \left[\frac{6\times5!}{1\times5!}\times(-96x^5)\right] + \left[\frac{6\times5\times4!}{2\times1\times4!}\times 144x^4\right] + \left[\frac{6\times5\times4\times3!}{3\times2\times1\times3!}\times(-216x^3)\right] +$$

$$\left[\frac{6\times5\times4!}{4!\times2!}\times 324x^2\right] + \left[\frac{6\times5!}{5!\times1!}\times(-486x)\right] + [1\times 729]$$

$$= 64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729$$

Q4. $\left(\frac{x}{3}+\frac{1}{x}\right)^5$

A.4. $\left(\frac{x}{3}+\frac{1}{x}\right)^5$

By using binomial theorem,

$$= \left[{}^5C_0\left(\frac{x}{3}\right)^5\right] + \left[{}^5C_1\left(\frac{x}{3}\right)^4\left(\frac{1}{x}\right)\right] + \left[{}^5C_2\left(\frac{x}{3}\right)^3\left(\frac{1}{x}\right)^2\right] + \left[{}^5C_3\left(\frac{x}{3}\right)^2\left(\frac{1}{x}\right)^3\right] + \left[{}^5C_4\left(\frac{x}{3}\right)\left(\frac{1}{x}\right)^4\right] + \left[{}^5C_5\left(\frac{1}{x}\right)^5\right]$$

$$= \left[\frac{5!}{0!\,(5-0)!}\times\frac{x^5}{243}\right] + \left[\frac{5!}{1!\,(5-1)!}\times\frac{x^4}{81}\times\frac{1}{x}\right] + \left[\frac{5!}{2!\,(5-2)!}\times\frac{x^3}{27}\times\frac{1}{x^2}\right] + \left[\frac{5!}{3!\,(5-3)!}\times\frac{x^2}{9}\times\frac{1}{x^3}\right] +$$

$$\left[\frac{5!}{4!\,(5-4)!}\times\frac{x}{3}\times\frac{1}{x^4}\right] + \left[\frac{5!}{5!\,(5-5)!}\times\frac{1}{x^5}\right]$$

$$= \left[1\times\frac{x^5}{243}\right] + \left[\frac{5\times4!}{1\times4!}\times\frac{x^3}{81}\right] + \left[\frac{5\times4\times3!}{2\times1\times3!}\times\frac{x}{27}\right] + \left[\frac{5\times4\times3!}{3!\times2!}\times\frac{1}{9x}\right] + \left[\frac{5\times4!}{4!\times1!}\times\frac{1}{3x^3}\right] + \left[1\times\frac{1}{x^5}\right]$$

$$= \frac{x^5}{243} + \frac{5x^3}{81} + \frac{10x}{27} + \frac{10}{9x} + \frac{5}{3x^3} + \frac{1}{x^5}$$

Q5. $\left(x+\frac{1}{x}\right)^6$

A.5. $\left(x+\frac{1}{x}\right)^6$

By binomial theorem,

$$= {}^6C_0(x^6) + {}^6C_1(x^4)\left(\frac{1}{x}\right) + {}^6C_2\;x^4\left(\frac{1}{x}\right)^2 + {}^6C_3(x^3)\left(\frac{1}{x}\right)^3 + {}^6C_4(x^2)\left(\frac{1}{x}\right)^4 + {}^6C_5(x)\left(\frac{1}{x}\right)^5 +$$

$${}^6C_6\left(\frac{1}{x}\right)^6$$

$$= \left[\frac{6!}{0!\,(6-0)!}\times x^6\right] + \left[\frac{6!}{1!\,(6-1)!}\times x^5\times\frac{1}{x}\right] + \left[\frac{6!}{2!\,(6-2)!}\times x^4\times\frac{1}{x^2}\right] + \left[\frac{6!}{3!\,(6-3)!}\times x^3\times\frac{1}{x^3}\right] +$$

$$\left[\frac{6!}{4!\,(6-4)!}\times x^2\times\frac{1}{x^4}\right] + \left[\frac{6!}{5!\,(6-5)!}\times x\times\frac{1}{x^5}\right] + \left[\frac{6!}{6!\,(6-6)!}\times\frac{1}{x^6}\right]$$

$$= [1\times x^6] + \left[\frac{6\times 5!}{1\times 5!}\times x^4\right] + \left[\frac{6\times5\times4!}{2\times1\times4!}\times x^2\right] + \left[\frac{6\times5\times4\times3!}{3\times2\times1\times3!}\times 1\right] + \left[\frac{6\times5\times4!}{4!\times2!}\times\frac{1}{x^2}\right] +$$

$$\left[\frac{6\times5!}{5!\times1!}\times\frac{1}{x^4}\right] + \left[1\times\frac{1}{x^6}\right]$$

$$= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$$

Using binomial theorem, evaluate each of the following:

Q6. $\mathbf{(96)^3}$

A.6. $(96)^3 = (100-4)^3$

Using $(x-y)^n$ expansion

$= {}^3C_0(100)^3 - {}^3C_1(100)^2(4) + {}^3C_2(100)(4)^2 - {}^3C_3(4)^3$

$$= \left[\frac{3!}{0!\,(3-0)!}\times 1000000\right] - \left[\frac{3!}{1!\,(3-1)!}\times 10000\times 4\right] + \left[\frac{3!}{2!\,(3-2)!}\times 100\times 16\right] - \left[\frac{3!}{3!\,(3-3)!}\times 64\right]$$

$$= [1\times 1000000] - \left[\frac{3\times 2!}{1\times 2!}\times 40000\right] + \left[\frac{3\times 2!}{2!\times 1!}\times 1600\right] - [1\times 64]$$

$= 1000000 - 120000 + 4800 - 64$

$= 1004800 - 120064$

$= 884736$

Q7. $\mathbf{(102)^5}$

A.7. $(102)^5 = (100+2)^5$

By using binomial theorem,

$= {}^5C_0(100)^5 + {}^5C_1(100)^4(2) + {}^5C_2(100)^3(2)^2 + {}^5C_3(100)^2(2)^3 + {}^5C_4(100)(2)^4 + {}^5C_5(2)^5$

$$= \left[\frac{5!}{0!\,(5-0)!}\times 10000000000\right] + \left[\frac{5!}{1!\,(5-1)!}\times 100000000\times 2\right] + \left[\frac{5!}{2!\,(5-2)!}\times 1000000\times 4\right] +$$

$$\left[\frac{5!}{3!\,(5-3)!}\times 10000\times 8\right] + \left[\frac{5!}{4!\,(5-4)!}\times 100\times 16\right] + \left[\frac{5!}{5!\,(5-5)!}\times 32\right]$$

$$= [1\times 10000000000] + \left[\frac{5\times 4!}{1\times 4!}\times 200000000\right] + \left[\frac{5\times4\times3!}{2\times1\times3!}\times 4000000\right] + \left[\frac{5\times4\times3!}{3!\times2!}\times 80000\right] +$$

$$\left[\frac{5\times 4!}{4!\times 1!}\times 1600\right] + [1\times 32]$$

$= 10000000000 + 1000000000 + 40000000 + 800000 + 8000 + 32$

$= 11040808032$

Q8. $\mathbf{(101)^4}$

A.8. $(101)^4 = (100 + 1)^4$

By binomial theorem we get,

$= {}^4C_0(100)^4 + {}^4C_1(100)^3(1) + {}^4C_2(100)^2(1)^2 + {}^4C_3(100)(1)^3 + {}^4C_4(1)^4$

$$= \left[\frac{4!}{0!\,(4-0)!}\times 100000000\right] + \left[\frac{4!}{1!\,(4-1)!}\times 1000000\times 1\right] + \left[\frac{4!}{2!\,(4-2)!}\times 10000\times 1\right] + \left[\frac{4!}{3!\,(4-3)!}\times 100\times 1\right] + \left[\frac{4!}{4!\,(4-4)!}\times 1\right]$$

$$= [1 \times 100000000] + \left[\frac{4\times 3!}{1\times 3!}\times 1000000\right] + \left[\frac{4\times 3\times 2!}{2\times 1\times 2!}\times 10000\right] + \left[\frac{4\times 3!}{3!\times 1!}\times 100\right] + [1 \times 1]$$

$= 100000000 + 4000000 + 60000 + 400 + 1$

$= 104060401$

Q9. $\mathbf{(99)^5}$

A.9. $(99)^5 = (100 - 1)^5$

By binomial theorem we get,

$= {}^5C_0(100)^5 - {}^5C_1(100)^4(1) + {}^5C_2(100)^3(1)^2 - {}^5C_3(100)^2(1)^3 + {}^5C_4(100)(1)^4 - {}^5C_5(1)^5$

$$= \left[\frac{5!}{0!\,(5-0)!}\times 10000000000\right] - \left[\frac{5!}{1!\,(5-1)!}\times 100000000\times 1\right] + \left[\frac{5!}{2!\,(5-2)!}\times 1000000\times 1\right] - \left[\frac{5!}{3!\,(5-3)!}\times 10000\times 1\right] + \left[\frac{5!}{4!\,(5-4)!}\times 100\times 1\right] - \left[\frac{5!}{5!\,(5-5)!}\times 1\right]$$

$$= [1 \times 10000000000] - \frac{5\times 4!}{1\times 4!}\times 100000000 + \left[\frac{5\times 4\times 3!}{2\times 1\times 3!}\times 100000\right] - \left[\frac{5\times 4\times 3!}{3!\times 2!}\times 10000\right] + \left[\frac{5\times 4!}{4!\times 1!}\times 100\right] - [1 \times 1]$$

$= 10000000000 - 500000000 + 10000000 - 100000 + 500 - 1$

$= 10010000500 - 500100001$

$= 9509900499$

Q10. **Using Binomial Theorem, indicate which number is larger $\mathbf{(1.1)^{10000}}$ or 1000.**

A.10. We know that,

$(1.1)^{10000} = (1 + 0.1)^{10000}$

By using binomial theorem,

$= {}^{10000}C_0(1)^{10000} + {}^{10000}C_1\,(1)^{(10000-1)}\,(0.1)$ + other positive terms

$$= \left[\frac{10000!}{0!\,(10000-0)!}\times 1\right] + \left[\frac{10000!}{1!\,(10000-1)!}\times 1\times 0.1\right] + \text{other positive terms}$$

$$= [1 \times 1] + \left[\frac{10000\times 9999!}{1\times 9999!}\times 0.1\right] + \text{other positive terms}$$

= 1 + 1000 + other positive terms

= 1001 + other positive terms

Hence, $(1.1)^{10000} > 1000$

Q11. Find $(a + b)^4 - (a - b)^4$. Hence, evaluate $\left(\sqrt{3}+\sqrt{2}\right)^4 - \left(\sqrt{3}-\sqrt{2}\right)^4$.

A.11. Using binomial theorem we have,

$(a + b)^4 - (a - b)^4$

$= [{}^4C_0a^4 + {}^4C_1a^3b + {}^4C_2a^2b^2 + {}^4C_3ab^3 + {}^4C_4b^4] - [{}^4C_0a^4 - {}^4C_1a^3b + {}^4C_2a^2b^2 - {}^4C_3ab^3 + {}^4C_4b^4]$

$= {}^4C_0a^4 + {}^4C_1a^3b + {}^4C_2a^2b^2 + {}^4C_3ab^3 + {}^4C_4b^4 - {}^4C_0a^4 + {}^4C_1a^3b - {}^4C_2a^2b^2 + {}^4C_3ab^3 - {}^4C_4b^4$

$= [2 \times {}^4C_1a^3b] + [2 \times {}^4C_3ab^3]$

$$= \left[2\times\frac{4!}{1!\,(4-1)!}a^3b\right] + \left[2\times\frac{4!}{3!\,(4-3)!}ab^3\right]$$

$$= \left[2\times\frac{4\times 3!}{1\times 3!}a^3b\right] + \left[2\times\frac{4\times 3!}{3!\times 1!}ab^3\right]$$

$= 8a^3b + 8ab^3$

Hence putting $a = \sqrt{3}$ and $b = \sqrt{2}$ we have,

$$\left(\sqrt{3}+\sqrt{2}\right)^4 - \left(\sqrt{3}-\sqrt{2}\right)^4$$

$$= 8\left(\sqrt{3}\right)^3\left(\sqrt{2}\right) + 8\left(\sqrt{3}\right)\left(\sqrt{2}\right)^3$$

$= (8 \times 3\sqrt{3} \times \sqrt{2}) + (8 \times \sqrt{3} \times 2\sqrt{2})$

$= 24\sqrt{6} + 16\sqrt{6}$

$= 40\sqrt{6}$

Q12. Find $(x + 1)^6 + (x - 1)^6$. Hence or otherwise evaluate $\left(\sqrt{2}+1\right)^6 + \left(\sqrt{2}-1\right)^6$.

A.12. By binomial expansion we have,

$(x+1)^6 + (x-1)^6$

$= [{}^6C_0(x^6) + {}^6C_1(x^5)(1) + {}^6C_2(x^4)(1)^2 + {}^6C_3(x^3)(1)^3 + {}^6C_4(x^2)(1)^4 + {}^6C_5(x)(1)^5 + {}^6C_6(1)^6] + [{}^6C_0(x^6) - {}^6C_1(x^5)(1) + {}^6C_2(x^4)(1)^2 - {}^6C_3(x^3)(1)^3 + {}^6C_4(x^2)(1)^4 - {}^6C_5(x)(1)^5 + {}^6C_6(1)^6]$

$= {}^6C_0(x^6) + {}^6C_1(x^5)(1) + {}^6C_2(x^4)(1)^2 + {}^6C_3(x^3)(1)^3 + {}^6C_4(x^2)(1)^4 + {}^6C_5(x)(1)^5 + {}^6C_6(1)^6 + {}^6C_0(x^6) - {}^6C_1(x^5)(1) + {}^6C_2(x^4)(1)^2 - {}^6C_3(x^3)(1)^3 + {}^6C_4(x^2)(1)^4 - {}^6C_5(x)(1)^5 + {}^6C_6(1)^6$

$= 2 \times [{}^6C_0(x^6) + {}^6C_2(x^4)(1)^2 + {}^6C_4(x^2)(1)^4 + {}^6C_6(1)^6]$

$$= 2 \times \left[\left(\frac{6!}{0!\,(6-0)!}\times x^6\right)+\left(\frac{6!}{2!\,(6-2)!}\frac{6!}{2!\,(6-2)!}+x^4\times 1\right) + \left(\frac{6!}{4!\,(6-4)!}\times x^2\times 1\right)+\left(\frac{6!}{6!\,(6-6)!}\times 1\right)\right]$$

$$= 2 \times [(1 \times x^6) + \left(\frac{6\times 5\times 4!}{2\times 1\times 4!}\times x^4\right)+\left(\frac{6\times 5\times 4!}{4!\times 2!}\times x^2\right) + (1 \times 1)]$$

$= 2[x^6 + 15x^4 + 15x^2 + 1]$

Hence putting x = $\sqrt{2}$ we have,

$(\sqrt{2} + 1)^6 + (\sqrt{2} - 1)^6$

$= 2 \times [\ \sqrt{2}^{\,6} + 15\ \sqrt{2}^{\,4} + 15\ \sqrt{2}^{\,2} + 1]$

$= 2 \times [2^3 + (15 \text{ x } 2^2) + (15 \text{ x } 2) + 1]$

$= 2 \times [8 + 60 + 30 + 1]$

$= 2 \times 99$

$= 198$

Q13. **Show that $9^{n+1} - 8n - 9$ is divisible by 64, whenever *n* is a positive integer.**

A.13. For a number x to be divisible by y, we can write x as a factor of y i.e., $x = ky$ where k is some natural number. Thus in order for $9^{n+1} - 8n - 9$ to be divisible by 64 we need to show that $9^{n+1} - 8n - 9 = 64k$ where k is some natural number.

We have, by binomial theorem

$(1 + a)^m = {}^mC_0 + {}^mC_1(a) + {}^mC_2(a)^2 + {}^mC_3(a)^3 + \ldots\ldots\ldots\ldots + {}^mC_m(a)^m$

Putting, $a = 8$ and $m = n + 1$

$(1 + 8)^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1.8 + {}^{n+1}C_2.8^2 + {}^{n+1}C_3.8^3 + \ldots\ldots + {}^{n+1}C_{n+1}.(8)^{n+1}$

$\Rightarrow 9^{n+1} = 1 + (n + 1)8 + 8^2 \times [{}^{n+1}C_2 + {}^{n+1}C_3.8 + \ldots\ldots\ldots + {}^{n+1}C_{n+1}.(8)^{n+1-2}]$ [since, ${}^{n+1}C_0 = 1$, ${}^{n+1}C_1 = n + 1$]

$\Rightarrow 9^{n+1} = 1 + 8n + 8 + 64 \times [{}^{n+1}C_2 + {}^{n+1}C_3.8 + \ldots\ldots\ldots + {}^{n+1}C_{n+1}.(8)^{n+1-2}]$

$\Rightarrow 9^{n+1} - 8n - 9 = 64 \times [{}^{n+1}C_2 + {}^{n+1}C_3.8 + \ldots\ldots\ldots + 8^{n-1}]$ [since, ${}^{n+1}C_{n+1} = 1$]

$\Rightarrow 9^{n+1} - 8n - 9 = 64k,$

where $k = {}^{n+1}C_2 + {}^{n+1}C_3.8 + \ldots\ldots\ldots + 8^{n-1}$ is a natural number.

This shows that $9^{n+1} - 8n - 9$ is divisible by 64.

Q14. Prove that $\sum_{r=0}^{n} 3^r\ {}^nC_r = 4^n$.

A.14. To prove, $\sum_{r=0}^{n} .3r\ .\ {}^nC_r = 4^n$

By binomial theorem,

$(a + b)^n = {}^nC_0(a)^n(b)^0 + {}^nC_1(a)^{n-1}(b)^1 + \ldots\ldots\ldots\ldots + {}^nC_r(a)^{n-r}(b)^r + \ldots\ldots\ldots\ldots + {}^nC_n(a)^{n-n}(b)^n$

Where, $b^0 = 1 = a^{n-n}$

So, $(a + b)^n = \sum_{r=0}^{n} .{}^nC_r(a)^{n-r}(b)^r$

Putting $a = 1$ and $b = 3$ such that $a + b = 4$, we can rewrite the above equation as

$(1 + 3)^n = \sum_{r=0}^{n} .{}^nC_r(1)^{n-r}.3^r$

$\Rightarrow 4^n = \sum_{r=0}^{n} .3r\ .{}^nC_r$

Hence proved.