MISCELLANEOUS EXERCISE

Question 1:

Find the value of k for which the line $(k-3)x-(4-k^2)y+k^2-7k+6=0$ is
(a) Parallel to x-axis.
(b) Parallel to y-axis.
(c) Passing through the origin.

Solution:

The given equation of the line is

$$(k-3)x-(4-k^2)y+k^2-7k+6=0 \qquad \ldots(1)$$

(a) If the given line is parallel to the x-axis then,
Slope of the given line = Slope of the x-axis
Then given line can be written as

$$(k-3)x+k^2-7k+6=(4-k^2)y$$

$$y=\frac{(k-3)}{(4-k^2)}x+\frac{k^2-7k+6}{(4-k^2)}$$

Which is of the form $y=mx+c$

Slope of the given line $=\frac{(k-3)}{(4-k^2)}$

Slope of the x-axis = 0

$$\Rightarrow \frac{k-4}{(4-k^2)}=0$$

$$\Rightarrow k-3=0$$

$$\Rightarrow k=3$$

Thus, the given line is parallel to x-axis, then the value of $k=3$.

(b) If the given line is parallel to the y-axis, it is vertical.
Hence, its slope will be undefined.

The slope of the given line is $=\frac{(k-3)}{(4-k^2)}$

Now, $\frac{(k-3)}{(4-k^2)}$ is defined at $k^2=4$

$$\Rightarrow k^2=4$$

$$\Rightarrow k=\pm 2$$

Thus, if the given line is parallel to the y-axis, then the value of $k=\pm 2$.

(c) if the given line is passing through the origin, then point $(0,0)$ satisfies the given equation of the line.

$$(k-3)(0)-(4-k^2)(0)+k^2-7k+6=0$$
$$k^2-7k+6=0$$
$$k^2-6k-k+6=0$$
$$(k-6)(k-1)=0$$
$$\Rightarrow k=6 \text{ or } k=1$$

Thus, if the given line is passing through the origin, then the values of k is either 1 or 6.

Question 2:

Find the values of θ and p, if the equation $x\cos\theta+y\sin\theta=p$ is the normal form of the line $\sqrt{3}x+y+2=0$

Solution:

The equation of the given line is $\sqrt{3}x+y+2=0$
This equation can be reduced as

$$\Rightarrow \sqrt{3}x+y+2=0$$
$$\Rightarrow -\sqrt{3}x-y=2$$

On dividing both sides by $\sqrt{(-\sqrt{3})^2+(-1)^2}=2$, we obtain

$$\Rightarrow -\frac{\sqrt{3}}{2}x-\frac{1}{2}y=\frac{2}{2}$$
$$\Rightarrow \left(-\frac{\sqrt{3}}{2}\right)x-\left(\frac{1}{2}\right)y=1 \qquad \ldots(1)$$

On comparing equation (1) to $x\cos\theta+y\sin\theta=p$, we obtain

$$\cos\theta=-\frac{\sqrt{3}}{2}, \sin\theta=-\frac{1}{2} \text{ and } p=1$$

Since the value of $\sin\theta$ and $\cos\theta$ are negative $\theta=\pi+\frac{\pi}{6}=\frac{7\pi}{6}$

Thus, the respective values of θ and p are $\frac{7\pi}{6}$ and 1.

Question 3:

Find the equation of the line, which cut-off intercepts on the axis whose sum and product are 1 and −6 respectively.

Solution:

Let the intercepts cut by the given lines on the axis be a and b.

It is given that

$$a+b=1 \qquad \ldots(1)$$

$$ab=-6 \qquad \ldots(2)$$

On solving equation (1) and (2), we obtain

$a=3$ and $b=-2$ or $a=-3$ and $b=3$

It is known that the equation of the line whose intercepts on the axis are a and b is

$\frac{x}{a}+\frac{y}{b}=1$ or $bx+ay-ab=0$

Case I: $a=3$ and $b=-2$

In this case, the equation of the line is $-2x+3y+6=0 \Rightarrow 2x-3y=6$

Case II: $a=-3$ and $b=3$

In this case, the equation of the line is $3x-2y+6=0 \Rightarrow -3x+2y=6$

Thus, the required equations of the lines are $2x-3y=6$ and $-3x+2y=6$

Question 4:

What are the points on the y-axis whose distance from the line $\frac{x}{3}+\frac{y}{4}=1$ is 4 units.

Solution:

Let $(0,b)$ be the point on y-axis whose distance from line $\frac{x}{3}+\frac{y}{4}=1$ is 4 units.

The given line can be written as

$$4x+3y-12=0 \qquad \ldots(1)$$

On comparing equation (1) to the general equation of line $Ax+By+C=0$, we obtain $A=4, B=-3$ and $C=-12$.

It is known that the perpendicular distance (d) of a line $Ax+By+C=0$ from a point (x_1,y_1) is given by $d=\frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}$

Therefore, if $(0,b)$ is the point on the y-axis whose distance from line $\frac{x}{3}+\frac{y}{4}=1$ is 4 units,

Then,

$$\Rightarrow 4=\frac{|4(0)+3(b)-12|}{\sqrt{4^2+3^2}}$$
$$\Rightarrow 4=\frac{|3b-12|}{5}$$
$$\Rightarrow 20=|3b-12|$$
$$\Rightarrow 20=\pm(3b-12)$$
$$\Rightarrow (3b-12)=20 \text{ or } (3b-12)=-20$$
$$\Rightarrow 3b=20+12 \text{ or } 3b=-20+12$$
$$\Rightarrow b=\frac{32}{3} \text{ or } b=\frac{-8}{3}$$

Thus, the required points are $\left(0,\frac{32}{3}\right)$ and $\left(0,-\frac{8}{3}\right)$

Question 5:

Find the perpendicular distance from the origin to the line joining the points $(\cos\theta, \sin\theta)$ and $(\cos\phi, \sin\phi)$.

Solution:

The equation of the line joining the points $(\cos\theta, \sin\theta)$ and $(\cos\phi, \sin\phi)$ is given by

$$\frac{y-\sin\theta}{x-\cos\theta}=\frac{\sin\phi-\sin\theta}{\cos\phi-\cos\theta}$$
$$y-\sin\theta=\frac{\sin\phi-\sin\theta}{\cos\phi-\cos\theta}(x-\cos\theta)$$

$$y(\cos\phi-\cos\theta)-\sin\theta(\cos\phi-\cos\theta)=x(\sin\phi-\sin\theta)-\cos\theta(\sin\phi-\sin\theta)$$
$$x(\sin\phi-\sin\theta)+y(\cos\phi-\cos\theta)+\cos\theta\sin\phi-\cos\theta\sin\theta-\sin\theta\cos\phi+\sin\theta\cos\theta=0$$
$$x(\sin\phi-\sin\theta)+y(\cos\phi-\cos\theta)+\sin(\phi-\theta)=0$$

$Ax+By+C=0$, where $A=\sin\theta-\sin\phi, B=\cos\phi-\cos\theta$ and $C=\sin(\phi-\theta)$

It is known that the perpendicular distance (d) of a line $Ax+By+C=0$ from a point (x_1,y_1) is given by $d=\frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}$

Therefore, the perpendicular distance (d) of the given line from point $(x_1,y_1)=(0,0)$ is

$$d = \frac{\left|(\sin\phi - \sin\theta)(0) + (\cos\phi - \cos\theta)(0) + \sin(\phi - \theta)\right|}{\sqrt{(\sin\phi - \sin\theta)^2 + (\cos\phi - \cos\theta)^2}}$$

$$= \frac{\left|\sin(\phi - \theta)\right|}{\sqrt{\sin^2\phi + \sin^2\theta - 2\sin\theta\sin\phi + \cos^2\phi + \cos^2\theta - 2\cos\phi\cos\theta}}$$

$$= \frac{\left|\sin(\phi - \theta)\right|}{\sqrt{(\sin^2\phi + \cos^2\phi) + (\sin^2\theta + \cos^2\theta) - 2(\sin\theta\sin\phi + \cos\phi\cos\theta)}}$$

$$= \frac{\left|\sin(\phi - \theta)\right|}{\sqrt{1 + 1 - 2(\cos(\phi - \theta))}}$$

$$= \frac{\left|\sin(\phi - \theta)\right|}{\sqrt{2(1 - \cos(\phi - \theta))}}$$

$$= \frac{\left|\sin(\phi - \theta)\right|}{\sqrt{2\left(2\sin^2\left(\frac{\phi - \theta}{2}\right)\right)}}$$

$$= \frac{\left|\sin(\phi - \theta)\right|}{\left|2\sin\left(\frac{\phi - \theta}{2}\right)\right|}$$

Where we know that $\sin\theta = 2\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)$

Then,

$$\frac{\left|2\cos\left(\frac{\phi - \theta}{2}\right)\sin\left(\frac{\phi - \theta}{2}\right)\right|}{\left|2\sin\left(\frac{\phi - \theta}{2}\right)\right|} = \left|\cos\left(\frac{\phi - \theta}{2}\right)\right|$$

Question 6:

Find the equation of the line parallel to y-axis and draw through the point of intersection of the lines $x - 7y + 5 = 0$ and $3x + y = 0$

Solution:

The equation of any line parallel to the y-axis is of the form

$$x = a \qquad \ldots(1)$$

The two given lines are

$$x - 7y + 5 = 0 \qquad \ldots(2)$$

$$3x + y = 0 \qquad \ldots(3)$$

On solving equation (2) and (3), we obtain $x=-\frac{5}{22}$ and $x=-\frac{15}{22}$

Therefore, $\left(-\frac{5}{22},-\frac{15}{22}\right)$ is the point of intersection of lines (2) and (3).

Since, line $x=a$ passes through point $\left(-\frac{5}{22},-\frac{15}{22}\right)$, $a=-\frac{5}{22}$

Thus, the required equation of the line is $x=-\frac{5}{22}$.

Question 7:

Find the equation of a line drawn perpendicular to the line $\frac{x}{4}+\frac{y}{6}=1$ through the point, where it meets the *y*-axis.

Solution:

The equation of the given line is $\frac{x}{4}+\frac{y}{6}=1$

This equation can also be written as $3x+2y-12=0$

$y=-\frac{3}{2}x+6$, which is of the form $y=mx+c$

Hence, the slope of the given line is $-\frac{3}{2}$

Slope of the line perpendicular to the given line is $-\frac{1}{\left(-\frac{3}{2}\right)}=\frac{2}{3}$

Let the given line intersect the *y*-axis at $(0,y)$.

On substituting $x=0$ in the equation of the given line, we obtain $\frac{y}{6}=1 \Rightarrow y=6$

The given line intersects the *y*-axis at $(0,6)$.

The equation of the line that has a slope of $\frac{2}{3}$ and passes through point $(0,6)$ is

$$(y-6)=\frac{2}{3}(x-0)$$
$$3y-18=2x$$
$$2x-3y+18=0$$

Thus, the required equation of the line is $2x-3y+18=0$.

Question 8:

Find the area of the triangle formed by the line $y-x=0, x+y=0$ and $x-k=0$.

Solution:

The equations of the given lines are:

$$y-x=0 \qquad \ldots(1)$$
$$x+y=0 \qquad \ldots(2)$$
$$x-k=0 \qquad \ldots(3)$$

The point of interaction of lines (1) and (2) is given by

$x=0$ and $y=0$.

The point of interaction of lines (2) and (3) is given by

$x=k$ and $y=-k$.

The point of interaction of lines (3) and (1) is given by

$x=k$ and $y=k$.

Thus, the vertices of the triangle formed by the three given lines are $(0,0),(k,-k)$ and $(k,-k)$.

We know that the area of a triangle whose vertices are $(x_1,y_1),(x_2,y_2)$ and (x_3,y_3) is

$$\frac{1}{2}\left|x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right|$$

Therefore, area of the triangle formed by three given lines

$$=\frac{1}{2}\left|0(-k-k)+k(k-0)+k(0+k)\right|$$
$$=\frac{1}{2}\left|k^2+k^2\right|$$
$$=\frac{1}{2}2k^2$$
$$=k^2$$

Hence, the area of the triangle is k^2 square units.

Question 9:

Find the value of p so that the three lines $3x+y-2=0,\ px+2y-3=0$ and $2x-y-3=0$ may intersect at one point.

Solution:

The equation of the given line are

$$3x + y - 2 = 0 \quad \ldots(1)$$
$$px + 2y - 3 = 0 \quad \ldots(2)$$
$$2x - y - 3 = 0 \quad \ldots(3)$$

On solving equations (1) and (3), we obtain

$$x = 1 \text{ and } y = -1$$

Since these three lines may intersect at one point, the point of intersection of lines (1) and (3) will also satisfy line (2)

$$p(1) + 2(-1) - 3 = 0$$
$$p - 2 - 3 = 0$$
$$p = 5$$

Thus, the required value of $p = 5$.

Question 10:

If three lines whose equations are $y_1 = m_1 x + c_1$, $y = m_2 x + c_2$ and $y = m_3 x + c_3$ are concurrent, then show that $m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0$.

Solution:

The equations of the given lines are:

$$y = m_1 x + c_1 \quad \ldots(1)$$
$$y = m_2 x + c_2 \quad \ldots(2)$$
$$y = m_3 x + c_3 \quad \ldots(3)$$

On subtracting equation (1) from (2) we obtain

$$(m_2 - m_1)x + (c_2 - c_1) = 0$$
$$(m_1 - m_2)x = (c_2 - c_1)$$
$$x = \frac{(c_2 - c_1)}{(m_1 - m_2)}$$

On substituting this value of x in (1), we obtain

$$y = m_1\left(\frac{c_2 - c_1}{m_1 - m_2}\right) + c_1$$

$$= \frac{m_1c_2 - m_1c_1}{m_1 - m_2} + c_1$$

$$= \frac{m_1c_2 - m_1c_1 + c_1(m_1 - m_2)}{m_1 - m_2}$$

$$= \frac{m_1c_2 - m_1c_1 + m_1c_1 - m_2c_1}{m_1 - m_2}$$

$$y = \frac{m_1c_2 - m_2c_1}{m_1 - m_2}$$

Therefore,

$\left(\frac{c_2 - c_1}{m_1 - m_2}, \frac{m_1c_2 - m_2c_1}{m_1 - m_2}\right)$ is the point of intersection of line (1) and (2)

It is given that lines (1), (2) and (3) are concurrent.

Hence the point of intersection of lines (1) and (2) will also satisfy equation (3).

On substituting this value of x and y in (3), we obtain

$$\frac{m_1c_2 - m_2c_1}{m_1 - m_2} = m_3\left(\frac{c_2 - c_1}{m_1 - m_2}\right) + c_3$$

$$\frac{m_1c_2 - m_2c_1}{m_1 - m_2} = \frac{m_3c_2 - m_3c_1 + c_3m_1 - c_3m_2}{m_1 - m_2}$$

$$m_1c_2 - m_2c_1 - m_3c_2 + m_3c_1 - m_1c_3 + m_2c_3 = 0$$

$$m_1c_2 - m_1c_3 - m_2c_1 + m_2c_3 - m_3c_2 + m_3c_1 = 0$$

$$m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0$$

Hence, $m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0$ proved.

Question 11:

Find the equation of the line through the points $(3,2)$ which make an angle of $45°$ with the line $x - 2y = 3$.

Solution:

Let the slope of the required line be m_1

The given line can be represented as $y = \frac{1}{2}x - \frac{3}{2}$, which is of the form $y = mx + c$

Slope of the given line $= m_2 = \frac{1}{2}$

It is given that the angle between the required line and line $x - 2y = 3$ is $45°$.

We know that if θ is the acute angle between lines l_1 and l_2 with the slopes m_1 and m_2 respectively,

Then,

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$

$$\tan 45° = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$

$$1 = \left|\frac{\frac{1}{2} - m_1}{1 + \frac{m_1}{2}}\right|$$

$$1 = \left|\frac{\frac{1 - 2m_1}{2}}{\frac{2 + m_1}{2}}\right|$$

$$1 = \left|\frac{1 - 2m_1}{2 + m_1}\right|$$

$$1 = \pm\left(\frac{1 - 2m_1}{2 + m_1}\right)$$

$$\Rightarrow 1 = \left(\frac{1 - 2m_1}{2 + m_1}\right) \qquad \Rightarrow 1 = -\left(\frac{1 - 2m_1}{2 + m_1}\right)$$

$$\Rightarrow 2 + m_1 = 1 - 2m_1 \qquad \Rightarrow 2 + m_1 = -1 + 2m_1$$

$$\Rightarrow m_1 = -\frac{1}{3} \quad \text{or} \quad \Rightarrow m_1 = 3$$

Case I: $m_1 = 3$

The equation of the line passing through $(3,2)$ and having a slope of 3 is:

$$y - 2 = 3(x - 3)$$
$$y - 2 = 3x - 9$$
$$3x - y = 7$$

Case I: $m_1 = -\frac{1}{3}$

The equation of the line passing through (3,2) and having a slope of $-\frac{1}{3}$ is

$$y-2=-\frac{1}{3}(x-2)$$
$$3y-6=-x+3$$
$$x+3y=9$$

Thus, the equations of the line are $3x-y=7$ and $x+3y=9$.

Question 12:

Find the equation of the line passing through the point of intersection of the line $4x+7y-3=0$ and $2x-3y+1=0$ that has equal intercepts on the axes.

Solution:

Let the equation of the line having equal intercepts on the axes be

$$\frac{x}{a}+\frac{y}{b}=1$$
$$\Rightarrow x+y=ab \qquad \ldots(1)$$

On solving equations $4x+7y-3=0$ and $2x-3y+1=0$, we obtain $x=\frac{1}{13}$ and $y=\frac{5}{13}$

Therefore,

$\left(\frac{1}{13},\frac{5}{13}\right)$ is the point of the intersection of the two given lines.

Since equation (1) passes through point $\left(\frac{1}{13},\frac{5}{13}\right)$

$$\frac{1}{13}+\frac{5}{13}=a$$
$$\Rightarrow a=\frac{6}{13}$$

Equation (1) becomes

$$\Rightarrow x+y=\frac{6}{13}$$
$$\Rightarrow 13x+13y=6$$

Thus, the required equation of the line $13x+13y=6$

Question 13:

Show that the equation of the line passing through the origin and making an angle θ with the line $y=mx+c$, is $\frac{y}{x}=\frac{m\pm\tan\theta}{1\mp m\tan\theta}$

Solution:

Let the equation of the line passing through the origin be $y = m_1 x$

If this line makes an angle of θ with line $y = mx + c$, then angle θ is given by

$$\tan\theta = \left|\frac{m_1 - m}{1 + m_1 m}\right|$$

$$= \left|\frac{\frac{y}{x} - m}{1 + \frac{y}{x}m}\right|$$

$$\tan\theta = \pm\left(\frac{\frac{y}{x} - m}{1 + \frac{y}{x}m}\right)$$

$$\Rightarrow \tan\theta = \frac{\frac{y}{x} - m}{1 + \frac{y}{x}m} \quad \text{or} \quad \tan\theta = -\left(\frac{\frac{y}{x} - m}{1 + \frac{y}{x}m}\right)$$

Case I: $\tan\theta = \frac{\frac{y}{x} - m}{1 + \frac{y}{x}m}$

$$\tan\theta = \frac{\frac{y}{x} - m}{1 + \frac{y}{x}m}$$

$$\tan\theta + \frac{y}{x}m\tan\theta = \frac{y}{x} - m$$

$$m + \tan\theta = \frac{y}{x}(1 - m\tan\theta)$$

$$\frac{y}{x} = \frac{m + \tan\theta}{1 - m\tan\theta}$$

Case II: $\tan\theta = -\left(\frac{\frac{y}{x} - m}{1 + \frac{y}{x}m}\right)$

$$\tan\theta = -\left(\frac{\frac{y}{x}-m}{1+\frac{y}{x}m}\right)$$

$$\tan\theta + \frac{y}{x}m\tan\theta = -\frac{y}{x}+m$$

$$m-\tan\theta = \frac{y}{x}(1+m\tan\theta)$$

$$\frac{y}{x} = \frac{m-\tan\theta}{1+m\tan\theta}$$

Thus, the required line is given by $\frac{y}{x} = \frac{m \pm \tan\theta}{1 \mp m\tan\theta}$

Question 14:

In what ratio, the joining $(-1,1)$ and $(5,7)$ is divisible by the line $x+y=4$?

Solution:

The solution of the line joining the points $(-1,1)$ and $(5,7)$ is given by

$$y-1 = \frac{7-1}{5+1}(x+1)$$

$$y-1 = \frac{6}{6}(x+1)$$

$$x-y+2=0 \qquad \ldots(1)$$

The equation of the line is

$$x+y=4 \qquad \ldots(2)$$

The points of intersection of line (1) and (2) is given by

$x=1$ and $y=3$

Let point $(1,3)$ divides the line segment joining $(-1,1)$ and $(5,7)$ in the ratio $1:k$.
Accordingly, by section formula

$$(1,3) = \left(\frac{k(-1)+1(5)}{1+k}, \frac{k(1)+1(7)}{1+k}\right)$$

$$\Rightarrow (1,3) = \left(\frac{-k+5}{1+k}, \frac{k+7}{1+k}\right)$$

$$\Rightarrow \frac{-k+5}{1+k} = 1, \ \frac{k+7}{1+k} = 3$$

Therefore,

$$\Rightarrow \frac{-k+5}{1+k} = 1$$
$$\Rightarrow -k+5 = 1+k$$
$$\Rightarrow 2k = 4$$
$$\Rightarrow k = 2$$

Thus, the line joining the points $(-1,1)$ and $(5,7)$ is divided by line $x+y=4$ in the ratio $1:2$.

Question 15:

Find the distance of the line $4x+7y+5=0$ from the point $(1,2)$ along the line $2x-y=0$.

Solution:

The given lines are

$$2x - y = 0 \qquad \ldots(1)$$
$$4x + 7y + 5 = 0 \qquad \ldots(2)$$

Let $A(1,2)$ is a point on the line (1) and B be the point intersection of line (1) and (2).

On solving equations (1) and (2), we obtain $x = -\frac{5}{18}$ and $y = -\frac{5}{9}$

Coordinates of point B are $\left(-\frac{5}{18}, -\frac{5}{9}\right)$

By using distance formula, the distance between points A and B can be obtained as

$$AB=\sqrt{\left(1+\frac{5}{18}\right)^2+\left(2+\frac{5}{9}\right)^2}$$

$$=\sqrt{\left(\frac{23}{18}\right)^2+\left(\frac{23}{9}\right)^2}$$

$$=\sqrt{\left(\frac{23}{9\times 2}\right)^2+\left(\frac{23}{9}\right)^2}$$

$$=\sqrt{\left(\frac{23}{9}\right)^2\left(\frac{1}{2}\right)^2+\left(\frac{23}{9}\right)^2}$$

$$=\sqrt{\left(\frac{23}{9}\right)^2\left(\frac{1}{4}+1\right)}$$

$$=\frac{23}{9}\sqrt{\left(\frac{5}{4}\right)}$$

$$=\frac{23}{9}\times\frac{\sqrt{5}}{2}$$

$$=\frac{23\sqrt{5}}{18}$$

Thus, the required distance is $\frac{23\sqrt{5}}{18}$ units.

Question 16:

Find the direction in which a straight line must be drawn through the points $(-1,2)$ so that its point of intersection with line $x-y=4$ may be at a distance of 3 units from this point.

Solution:

Let $y=mx+c$ be the line through point $(-1,2)$
Accordingly,

$$\Rightarrow 2=m(-1)+c$$

$$\Rightarrow 2=-m+c$$

$$\Rightarrow c=m+2$$

$$\Rightarrow y=mx+m+2 \qquad \ldots(1)$$

The given line is

$$x-y=4 \qquad \ldots(2)$$

On solving equation (1) and (2), we obtain

$$x=\frac{2-m}{m+1} \text{ and } y=\frac{5m+2}{m+1}$$

Therefore,

$\left(\frac{2-m}{m+1},\frac{5m+2}{m+1}\right)$ is the point of intersection of line (1) and (2)

Since this point is at a distance of 3 units from point $(-1,2)$, accordingly to distance formula,

$$\Rightarrow \sqrt{\left(\frac{2-m}{m+1}+1\right)^2+\left(\frac{5m+2}{m+1}-2\right)^2}=3$$

$$\Rightarrow \left(\frac{2-m+m+1}{m+1}\right)^2+\left(\frac{5m+2-2m-2}{m+1}\right)^2=3^2$$

$$\Rightarrow \frac{9}{(m+1)^2}+\frac{9m^2}{(m+1)^2}=9$$

$$\Rightarrow \frac{1+m^2}{(m+1)^2}=1$$

$$\Rightarrow 1+m^2=m^2+1+2m$$

$$\Rightarrow 2m=0$$

$$\Rightarrow m=0$$

Thus, the slope of the required line must be zero i.e., the line must be parallel to the *x*-axis.

Question 17:

The hypotenuse of a right-angled triangle has its end at the points $(1,3)$ and $(-4,1)$. Find the equation of the legs (perpendicular sides) of the triangle.

Solution:

Let ABC be the right-angled triangle, where $\angle C=90°$

Let the slope of $AC=m$

Hence, the slope of $BC=-\frac{1}{m}$

Equation of AC:

$$\Rightarrow y-3=m(x-1)$$

$$\Rightarrow (x-1)=\frac{y-3}{m}$$

Equation of BC:

$$x+4=-m(y-1)$$

For a given value of m, we can get these equations

For, $m = 0,\ y - 3 = 0;\ x + 4 = 0$

For $m \to \infty,\ x - 1 = 0;\ y - 1 = 0$

Question 18:

Find the image of the point $(3,8)$ with respect to the line $x + 3y = 7$ assuming the line to be a plane mirror.

Solution:

The equation of the given line is

$$x + 3y = 7 \qquad \ldots(1)$$

Let point $B(a,b)$ be the image of point $A(3,8)$
Accordingly, line (1) is the perpendicular bisector of AB

Slope of $AB = \dfrac{b-8}{a-3}$, while the slope of the line (1) is $-\dfrac{1}{3}$

Since line (1) is perpendicular to AB

$$\Rightarrow \left(\frac{b-8}{a-3}\right)\left(-\frac{1}{3}\right) = 1$$

$$\Rightarrow \frac{b-8}{3a-9} = 1$$

$$\Rightarrow b - 8 = 3a - 9$$

$$\Rightarrow 3a - b - 1$$

Mid-point of $AB = \left(\dfrac{a+3}{2}, \dfrac{b+8}{2}\right)$

The mid-point of the line segments AB will also satisfy line (1).

Hence, from equation (1), we have

$$\Rightarrow \left(\frac{a+3}{2}\right) + 3\left(\frac{b+8}{2}\right) = 7$$

$$\Rightarrow a + 3 + 3b + 24 = 14$$

$$\Rightarrow a + 3b = -13 \ldots\ldots(3)$$

On solving equations (2) and (3), we obtain

$$a=-1 \text{ and } b=-4.$$

Thus, the image of the given point with respect to the given line is $(-1,-4)$.

Question 19:

If the lines $y=3x+1$ and $2y=x+3$ are equally indicated to the line $y=mx+4$, find the value of m.

Solution:

The equations of the given lines are:

$$y=3x+1 \quad \ldots(1)$$
$$2y=x+3 \quad \ldots(2)$$
$$y=mx+4 \quad \ldots(3)$$

Slope of line (1), $m_1=3$

Slope of line (2), $m_2=\frac{1}{2}$

Slope of line (3), $m_3=m$

It is given that lines (1) and (2) are equally inclined to line (3). This means that the given angle between lines (1) and (3) equals the angle between lines (2) and (3).

Therefore,

$$\Rightarrow \left|\frac{m_1-m_3}{1+m_1m_3}\right|=\left|\frac{m_2-m_3}{1+m_2m_3}\right|$$

$$\Rightarrow \left|\frac{3-m}{1+3m}\right|=\left|\frac{\frac{1}{2}-m}{1+\frac{1}{2}m}\right|$$

$$\Rightarrow \left|\frac{3-m}{1+3m}\right|=\left|\frac{1-2m}{m+2}\right|$$

$$\Rightarrow \left|\frac{3-m}{1+3m}\right|=\pm\left(\frac{1-2m}{m+2}\right)$$

$$\Rightarrow \frac{3-m}{1+3m}=\frac{1-2m}{m+2} \, or \, \frac{3-m}{1+3m}=-\left(\frac{1-2m}{m+2}\right)$$

<u>Case I</u>: If $\frac{3-m}{1+3m}=\frac{1-2m}{m+2}$

Then,

$$\Rightarrow (3-m)(m+2)=(1-2m)(1+3m)$$
$$\Rightarrow -m^2+m-6=1+m-6m^2$$
$$\Rightarrow 5m^2+5=0$$
$$\Rightarrow m^2+1=0$$
$$\Rightarrow m^2=-1$$
$$\Rightarrow m=\sqrt{-1}$$

Here, $m=\sqrt{-1}$, which is not real
Hence, this case is not possible

Case II: If $\frac{3-m}{1+3m}=-\left(\frac{1-2m}{m+2}\right)$
Then,

$$\Rightarrow (3-m)(m+2)=-(1-2m)(1+3m)$$
$$\Rightarrow -m^2+m-6=-(1+m-6m^2)$$
$$\Rightarrow 7m^2-2m-7=0$$
$$\Rightarrow m=\frac{-2\pm\sqrt{4-4(7)(-7)}}{2(7)}$$
$$\Rightarrow m=\frac{2\pm2\sqrt{1+49}}{14}$$
$$\Rightarrow m=\frac{1\pm5\sqrt{2}}{7}$$

Thus, the required value of $m=\frac{1\pm5\sqrt{2}}{7}$.

Question 20:

If sum of the perpendicular distance of a variable point $P(x,y)$ from the lines $x+y-5=0$ and $3x-2y+7=0$ is always 10. Show that P must move on a line.

Solution:

The equations of the lines are

$$x+y-5=0 \qquad \ldots(1)$$
$$3x-2y+7=0 \qquad \ldots(2)$$

The perpendicular distance of $P(x,y)$ from lines (1) and (2) are respectively given by

$$d_1 = \frac{|x+y-5|}{\sqrt{(1)^2+(1)^2}} \text{ and } d_2 = \frac{|3x-2y+7|}{\sqrt{(3)^2+(-2)^2}}$$

i.e., $d_1 = \frac{|x+y-5|}{\sqrt{2}}$ and $d_2 = \frac{|3x-2y+7|}{\sqrt{13}}$

It is given that $d_1 + d_2 = 10$
Therefore,

$$\Rightarrow \frac{|x+y-5|}{\sqrt{2}} + \frac{|3x-2y+7|}{\sqrt{13}} = 10$$

$$\Rightarrow \sqrt{13}\,|x+y-5| + \sqrt{2}\,|3x-2y+7| - 10\sqrt{26} = 0$$

$$\Rightarrow \sqrt{13}(x+y-5) + \sqrt{2}(3x-2y+7) - 10\sqrt{26} = 0$$

Assuming $x+y-5=0$ and $3x-2y+7=0$ are positive.

$$\Rightarrow \sqrt{13}x + \sqrt{13}y - 5\sqrt{13} + 3\sqrt{2}x - 2\sqrt{2}y + 7\sqrt{2} - 10\sqrt{26} = 0$$

$$\Rightarrow (\sqrt{13}+3\sqrt{2})x + (\sqrt{13}-2\sqrt{2})y + (7\sqrt{2}-5\sqrt{13}-10\sqrt{26}) = 0$$

Since, $(\sqrt{13}+3\sqrt{2})x + (\sqrt{13}-2\sqrt{2})y + (7\sqrt{2}-5\sqrt{13}-10\sqrt{26}) = 0$ is the equation of a line.

Similarly, we can obtain the equation of line for any signs of $x+y-5=0$ and $3x-2y+7=0$.

Thus, point P must move on a line.

Question 21:

Find equation of the line which is equidistant from parallel lines $9x+6y-7=0$ and $3x+2y+6=0$.

Solution:
The equations of the given lines are

$$9x+6y-7=0 \quad \ldots(1)$$

$$3x+2y+6=0 \quad \ldots(2)$$

Let $P(h,k)$ be the arbitrary point, is equidistant from lines (1) and (2).

Then the perpendicular distance of $P(h,k)$ from the line (1) is given by

$$d_1=\frac{|9h+6k-7|}{\sqrt{(9)^2+(6)^2}}$$
$$=\frac{|9h+6k-7|}{\sqrt{117}}$$
$$=\frac{|9h+6k-7|}{3\sqrt{13}}$$

And the perpendicular distance of $P(h,k)$ from line (2) is given by

$$d_2=\frac{|3h+2k+6|}{\sqrt{(3)^2+(2)^2}}$$
$$=\frac{|3h+2k+6|}{\sqrt{13}}$$

Since $P(h,k)$ is equidistant from lines (1) and (2), $d_1=d_2$

Therefore,

$$\Rightarrow\frac{|9h+6k-7|}{3\sqrt{13}}=\frac{|3h+2k+6|}{\sqrt{13}}$$
$$\Rightarrow|9h+6k-7|=3|3h+2k+6|$$
$$\Rightarrow 9h+6k-7=\pm3(3h+2k+6)$$

<u>Case I</u>: $9h+6k-7=3(3h+2k+6)$

$$\Rightarrow 9h+6k-7=3(3h+2k+6)$$
$$\Rightarrow 9h+6k-7=9h+6k+18$$
$$\Rightarrow 9h+6k-7-9h-6k-18=0$$
$$\Rightarrow 25=0$$

Which is an absurd, hence this case is not possible.

<u>Case II</u>: $9h+6k-7=-3(3h+2k+6)$

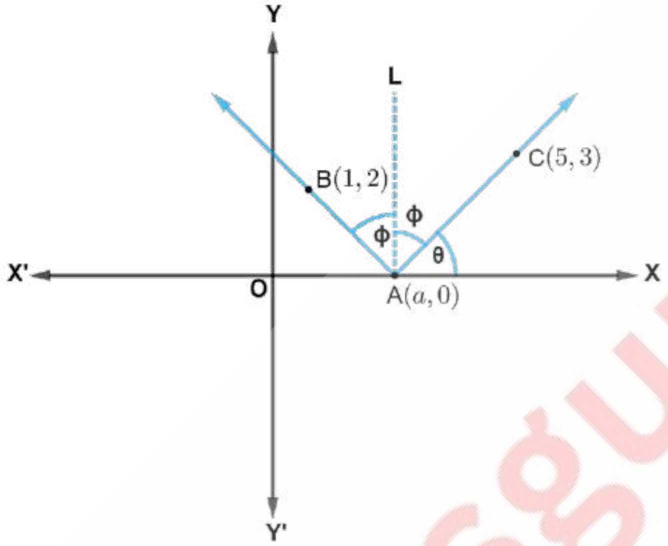
$$\Rightarrow 9h+6k-7=-3(3h+2k+6)$$
$$\Rightarrow 9h+6k-7=-9h-6k-18$$
$$\Rightarrow 18h+12k+11=0$$

Thus, the required equation of the line is $18x+12y+11=0$.

Question 22:

A ray of light passing through the point $(1,2)$ reflects on the x-axis at point A and the reflected ray passes through the point $(5,3)$. Find the coordinates of A.

Solution:



Let the coordinates of point A be $(a,0)$.
Draw a line, AL perpendicular to the x-axis

We know that angle of incidence is equal to angle of reflection.

Hence, let $\angle BAL = \angle CAL = \phi$ and $\angle CAX = \theta$
Now,

$$\begin{aligned}\angle OAB &= 180^\circ - (\theta + 2\phi) \\ &= 180^\circ - [\theta + 2(90^\circ - \theta)] \\ &= 180^\circ - [\theta + 180^\circ - 2\theta] \\ &= 180^\circ - 180^\circ + \theta \\ &= \theta\end{aligned}$$

Therefore,

$$\angle BAX = 180^\circ - \theta$$

Now,

Slope of line $AC = \frac{3-0}{5-a}$

$$\Rightarrow \tan\theta = \frac{3-0}{5-a} \qquad \ldots(1)$$

Slope of line $AC = \frac{2-0}{1-a}$

$$\Rightarrow \tan(180° - \theta) = \frac{2}{1-a}$$

$$\Rightarrow -\tan\theta = \frac{2}{1-a}$$

$$\Rightarrow \tan\theta = \frac{2}{a-1} \qquad \ldots(2)$$

From equations (1) and (2), we obtain

$$\Rightarrow \frac{3}{5-a} = \frac{2}{a-1}$$

$$\Rightarrow 3a - 3 = 10 - 2a$$

$$\Rightarrow a = \frac{13}{5}$$

Thus, the coordinates of point A are $\left(\frac{13}{5}, 0\right)$.

Question 23:

Prove that the product of the lengths of the perpendiculars drawn from the points $\left(\sqrt{a^2-b^2}, 0\right)$ and $\left(-\sqrt{a^2+b^2}, 0\right)$ to the line $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ is b^2.

Solution:

The equation of the given line is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

$$bx\cos\theta + ay\sin\theta - ab = 0 \qquad \ldots(1)$$

Length of the perpendicular from point $\left(\sqrt{a^2+b^2}, 0\right)$ to the line (1) is

$$p_1 = \frac{\left|b\cos\theta\left(\sqrt{a^2-b^2}\right) + a\sin\theta(0) - ab\right|}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

$$= \frac{\left|b\cos\theta\left(\sqrt{a^2-b^2}\right) - ab\right|}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} \qquad \ldots(2)$$

Length of the perpendicular from point $\left(-\sqrt{a^2+b^2}, 0\right)$ to the line (2) is

$$p_2 = \frac{\left|b\cos\theta\left(-\sqrt{a^2-b^2}\right)+a\sin\theta(0)-ab\right|}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}$$

$$= \frac{\left|b\cos\theta\left(\sqrt{a^2-b^2}\right)-ab\right|}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}} \quad \ldots(3)$$

On multiplying equations (2) and (3), we obtain

$$p_1p_2 = \frac{\left|b\cos\theta\left(\sqrt{a^2-b^2}\right)-ab\right|\left|b\cos\theta\left(\sqrt{a^2-b^2}\right)-ab\right|}{\left(\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}\right)^2}$$

$$= \frac{\left|\left(b\cos\theta\left(\sqrt{a^2-b^2}\right)-ab\right)\left(b\cos\theta\left(\sqrt{a^2-b^2}\right)-ab\right)\right|}{\left(b^2\cos^2\theta+a^2\sin^2\theta\right)}$$

$$= \frac{\left|\left(b\cos\theta\sqrt{a^2-b^2}\right)^2-(ab)^2\right|}{\left(b^2\cos^2\theta+a^2\sin^2\theta\right)}$$

$$= \frac{\left|b^2\cos^2\theta\left(a^2-b^2\right)-a^2b^2\right|}{\left(b^2\cos^2\theta+a^2\sin^2\theta\right)}$$

$$= \frac{\left|a^2b^2\cos^2\theta-b^4\cos^2\theta-a^2b^2\right|}{\left(b^2\cos^2\theta+a^2\sin^2\theta\right)}$$

$$= \frac{b^2\left|a^2\cos^2\theta-b^2\cos^2\theta-a^2\right|}{\left(b^2\cos^2\theta+a^2\sin^2\theta\right)}$$

$$= \frac{b^2\left|a^2\cos^2\theta-b^2\cos^2\theta-a^2\sin^2\theta-a^2\cos^2\theta\right|}{\left(b^2\cos^2\theta+a^2\sin^2\theta\right)} \quad \left[\because \sin^2\theta+\cos^2\theta=1\right]$$

$$= \frac{b^2\left|-\left(b^2\cos^2\theta+a^2\sin^2\theta\right)\right|}{\left(b^2\cos^2\theta+a^2\sin^2\theta\right)}$$

$$= \frac{b^2\left(b^2\cos^2\theta+a^2\sin^2\theta\right)}{\left(b^2\cos^2\theta+a^2\sin^2\theta\right)}$$

$$= b^2$$

Hence, proved

Question 24:

A person standing at the junction (crossing) of two straight paths represented by the equation $2x-3y+4=0$ and $3x+4y-5=0$ wants to reach the path whose equation is $6x-7y+8=0$ in the least time. Find equation of the path that he should follow.

Solution:

The equations of the given lines are

$$2x-3y+4=0 \quad \ldots(1)$$
$$3x+4y-5=0 \quad \ldots(2)$$
$$6x-7y+8=0 \quad \ldots(3)$$

The person is standing at the junction of the paths represented by lines (1) and (2).

On solving equations (1) and (2), we obtain $x=-\frac{1}{17}$ and $y=\frac{22}{17}$

Thus, the person is standing at point $\left(-\frac{1}{17},\frac{22}{17}\right)$

The person can reach path (3) in the least time if he walks along the perpendicular line to (3) from point $\left(-\frac{1}{17},\frac{22}{17}\right)$

Now,

Slope of the line (3) $=\frac{6}{7}$

Slope of the line perpendicular to line (3) $=-\frac{1}{\left(\frac{6}{7}\right)}=-\frac{7}{6}$

The equation of the line passing through $\left(-\frac{1}{17},\frac{22}{17}\right)$ and having a slope of $-\frac{7}{6}$ is given by

$$\Rightarrow\left(y-\frac{22}{17}\right)=-\frac{7}{6}\left(x+\frac{1}{17}\right)$$
$$\Rightarrow 6(17y-22)=-7(17x+1)$$
$$\Rightarrow 102y-132=-119x-7$$
$$\Rightarrow 119x+102y=125$$

Hence, the path that the person should follow is $119x+102y=125$.