Exercise 2.3

1. *Determine which of the following polynomials has* $(x + 1)$ a *factor*:

   (*i*) $x^3 + x^2 + x + 1$ (*ii*) $x^4 + x^3 + x^2 + x + 1$

   (*iii*) $x^4 + 3x^3 + 3x^2 + x + 1$ (*iv*) $x^3 - x^2 - (2 + \sqrt{2})x + \sqrt{2}$

Sol. (*i*) If $(x + 1)$ is a factor of $p(x) = x^3 + x^2 + x + 1$, then $p(-1) = 0$

Now, $p(-1) = -1 + 1 - 1 + 1 = 0$.

Hence, $(x + 1)$ is a factor.

(*ii*) If $(x + 1)$ is a factor of $p(x) = x^4 + x^3 + x^2 + x + 1$, then $p(-1) = 0$.

Now, $p(-1) = 1 - 1 + 1 - 1 + 1 = 1 \neq 0$.

Hence, $(x + 1)$ is not a factor.

(*iii*) If $(x + 1)$ is a factor of $p(x) = x^4 + 3x^3 + 3x^2 + x + 1$, then $p(-1) = 0$.

Now, $p(-1) = 1 - 3 + 3 - 1 + 1 = 1 \neq 0$.

Hence, $(x + 1)$ is not a factor.

(*iv*) If $(x + 1)$ is a factor of $p(x) = x^3 - x^2 - (2 + \sqrt{2})x + \sqrt{2}$, then $p(-1) = 0$.

Now, $p(-1) = (-1)^3 - (-1)^2 - (2 + \sqrt{2})(-1) + \sqrt{2}$

$= -1 - 1 + 2 + \sqrt{2} + \sqrt{2}$

$= 2\sqrt{2} \neq 0$.

Hence, $(x + 1)$ is not a factor.

2. *Use the Factor Theorem to determine whether g(x) is a factor of p(x) in each of the following cases:*

(*i*) $p(x) = 2x^3 + x^2 - 2x - 1$, $g(x) = x + 1$

(*ii*) $p(x) = x^3 + 3x^2 + 3x + 1$, $g(x) = x + 2$

(*iii*) $p(x) = x^3 - 4x^2 + x + 6$, $g(x) = x - 3$.

Sol. (*i*) If $g(x)$ is a factor of $p(x)$, then $p(-1) = 0$.

Now, $p(-1) = -2 + 1 + 2 - 1 = 0$.

Hence, $g(x)$ is a factor of $p(x)$.

(*ii*) If $g(x)$ is a factor of $p(x)$, then $p(-2) = 0$.

Now, $p(-2) = -8 + 12 - 6 + 1 = -14 + 13 = -1 \neq 0$.

Hence, $g(x)$ is not a factor of $p(x)$.

(*iii*) If $g(x)$ is a factor of $p(x)$, then $p(3) = 0$.

Now, $p(3) = 27 - 36 + 3 + 6 = 36 - 36 = 0$.

Hence, $g(x)$ is a factor of $p(x)$.

3. *Find the value of k, if x – 1 is a factor of p(x) in each of the following cases:*

(*i*) $p(x) = x^2 + x + k$ (*ii*) $p(x) = 2x^2 + kx + \sqrt{2}$

(*iii*) $p(x) = kx^2 - \sqrt{2}x + 1$ (*iv*) $p(x) = kx^2 - 3x + k$.

Sol. (*i*) If $(x - 1)$ is a factor of $p(x)$, then $p(1) = 0$.

Now, $p(1) = 0 \Rightarrow 1 + 1 + k = 0 \Rightarrow k = -2$.

(*ii*) If $(x - 1)$ is a factor of $p(x)$, then $p(1) = 0$.

Now, $p(1) = 0 \Rightarrow 2 + k + \sqrt{2} = 0 \Rightarrow k = -(2 + \sqrt{2})$.

(*iii*) If $(x - 1)$ is a factor of $p(x)$, then $p(1) = 0$.

Now, $p(1) = 0 \Rightarrow k - \sqrt{2} + 1 = 0 \Rightarrow k = \sqrt{2} - 1$.

(*iv*) If $(x - 1)$ is a factor of $p(x)$, then $p(1) = 0$.

Now, $p(1) = 0 \Rightarrow k - 3 + k = 0 \Rightarrow k = \frac{3}{2}$.

4. *Factorise*:

(*i*) $12x^2 - 7x + 1$ (*ii*) $2x^2 + 7x + 3$

(*iii*) $6x^2 + 5x - 6$ (*iv*) $3x^2 - x - 4$.

Sol. (*i*) $12x^2 - 7x + 1 = 12x^2 - 4x - 3x + 1$

$= 4x(3x - 1) - 1(3x - 1) = (4x - 1)(3x - 1)$

(*ii*) $2x^2 + 7x + 3 = 2x^2 + 6x + x + 3$

$= 2x(x + 3) + 1(x + 3) = (2x + 1)(x + 3)$.

(*iii*) $6x^2 + 5x - 6 = 6x^2 + 9x - 4x - 6$

$= 3x(2x + 3) - 2(2x + 3) = (3x - 2)(2x + 3)$.

(*iv*) $3x^2 - x - 4 = 3x^2 - 4x + 3x - 4$

$= x(3x - 4) + 1(3x - 4) = (x + 1)(3x - 4)$.

5. *Factorise*:

(*i*) $x^3 - 2x^2 - x + 2$ (*ii*) $x^3 - 3x^2 - 9x - 5$

(*iii*) $x^3 + 13x^2 + 32x + 20$ (*iv*) $2y^3 + y^2 - 2y - 1$.

Sol. (*i*) Let $p(x) = x^3 - 2x^2 - x + 2$.

Possible factors of 2 are $\pm 1, \pm 2$.

We notice $p(1) = 1 - 2 - 1 + 2 = 0$

$\Rightarrow$ $x = 1$ is a zero of polynomial $p(x)$ or $(x - 1)$ is a factor of $p(x)$.

Let us divide $p(x)$ by $(x - 1)$.

$$\begin{array}{r|l} x-1 \;\overline{)\; x^3 - 2x^2 - x + 2} \quad (x^2 - x - 2 & \\ \underline{x^3 - x^2} & \\ - \quad + & \\ -x^2 - x & \\ \underline{-x^2 + x} & \\ + \quad - & \\ -2x + 2 & \\ \underline{-2x + 2} & \\ + \quad - & \\ 0 & \end{array}$$

First term of quotient $= \frac{x^3}{x} = x^2$

Second term of quotient $= \frac{-x^2}{x} = -x$

Third term of quotient $= \frac{-2x}{x} = -2$

$$\therefore \quad p(x) = x^3 - 2x^2 - x + 2$$
$$= (x - 1)(x^2 - x - 2) = (x - 1)(x^2 - 2x + x - 2)$$
$$= (x - 1)\{x(x - 2) + 1(x - 2)\}$$
$$= (x - 1)(x + 1)(x - 2).$$

Alternative Method:

$$x^3 - 2x^2 - x + 2 = x^2(x - 2) - 1(x - 2)$$
$$= (x - 2)(x^2 - 1)$$
$$= (x - 2)(x^2 - 1^2)$$
$$= (x - 2)(x + 1)(x - 1).$$

(*ii*) Let $p(x) = x^3 - 3x^2 - 9x - 5$.

Possible factors of 5 are $\pm 1, \pm 5$

We notice $p(-1) = -1 - 3 + 9 - 5 = 0$

$\Rightarrow$ $x = -1$ is zero of polynomial $p(x)$.

$\Rightarrow$ $(x + 1)$ is a factor of $p(x)$.

Let us divide $p(x)$ by $(x + 1)$.

$$\begin{array}{r|l} x+1 & x^3 - 3x^2 - 9x - 5 \quad (x^2 - 4x - 5 \\ & \underline{x^3 + x^2} \\ & \quad - \quad - \\ & -4x^2 - 9x \\ & \underline{-4x^2 - 4x} \\ & \quad + \quad + \\ & -5x - 5 \\ & \underline{-5x - 5} \\ & \quad + \quad + \\ & 0 \end{array}$$

First term of quotient

$$= \frac{x^3}{x} = x^2$$

Second term of quotient

$$= \frac{-4x^2}{x} = -4x$$

Third term of quotient

$$= \frac{-5x}{x} = -5$$

$$\therefore \quad p(x) = x^3 - 3x^2 - 9x - 5$$
$$= (x + 1)(x^2 - 4x - 5)$$
$$= (x + 1)(x^2 - 5x + x - 5)$$
$$= (x + 1)\{x(x - 5) + 1(x - 5)\}$$
$$= (x + 1)(x + 1)(x - 5).$$

(*iii*) Let $p(x) = x^3 + 13x^2 + 32x + 20$.

Possible factors of 20 are $\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20$.

We notice $p(-1) = -1 + 13 - 32 + 20 = 0$

$\Rightarrow \quad x = -1$ is a zero of $p(x)$

$\Rightarrow \quad (x + 1)$ is a factor of $p(x)$.

Let us divide $p(x)$ by $(x + 1)$.

$$
\begin{array}{r|l}
\begin{array}{r}
x^2 + 12x + 20 \\
x+1 \overline{\big) x^3 + 13x^2 + 32x + 20} \\
x^3 + x^2 \qquad\qquad\quad \\
- \quad - \qquad\qquad\quad \\
\hline
12x^2 + 32x \qquad \\
12x^2 + 12x \qquad \\
- \quad - \qquad \\
\hline
20x + 20 \\
20x + 20 \\
- \quad - \\
\hline
0 \\
\hline
\end{array}
&
\begin{array}{l}
\text{First term of quotient} \\
= \dfrac{x^3}{x} = x^2 \\
\text{Second term of quotient} \\
= \dfrac{12x^2}{x} = 12x \\
\text{Third term of quotient} \\
= \dfrac{20x}{x} = 20
\end{array}
\end{array}
$$

$$
\begin{aligned}
\therefore \quad p(x) &= x^3 + 13x^2 + 32x + 20 \\
&= (x + 1)(x^2 + 12x + 20) \\
&= (x + 1)(x^2 + 10x + 2x + 20) \\
&= (x + 1)\{x(x + 10) + 2(x + 10)\} \\
&= (x + 1)(x + 2)(x + 10).
\end{aligned}
$$

(*iv*) Let $p(y) = 2y^3 + y^2 - 2y - 1$

$$
\begin{aligned}
&= y^2(2y + 1) - 1(2y + 1) = (y^2 - 1)(2y + 1) \\
&= (y - 1)(y + 1)(2y + 1).
\end{aligned}
$$