Exercise 1.2

Question 1:

Prove that $\sqrt{5}$ is irrational.

Answer:

Let $\sqrt{5}$ is a rational number.

Therefore, we can find two integers *a*, *b* (*b* ≠ 0) such that $\sqrt{5} = \frac{a}{b}$

Let *a* and *b* have a common factor other than 1. Then we can divide them by the common factor, and assume that *a* and *b* are co-prime.

$$a = \sqrt{5}b$$
$$a^2 = 5b^2$$

Therefore, a^2 is divisible by 5 and it can be said that *a* is divisible by 5.

Let *a* = 5*k*, where *k* is an integer

$$(5k)^2 = 5b^2$$
$$b^2 = 5k^2$$

This means that b^2 is divisible by 5 and hence, *b* is divisible by 5.

This implies that *a* and *b* have 5 as a common factor.

And this is a contradiction to the fact that *a* and *b* are co-prime.

Hence, $\sqrt{5}$ cannot be expressed as $\frac{p}{q}$ or it can be said that $\sqrt{5}$ is irrational.

Question 2:

Prove that $3+2\sqrt{5}$ is irrational.

Answer:

Let $3+2\sqrt{5}$ is rational.

Therefore, we can find two integers *a*, *b* (*b* ≠ 0) such that

$$3+2\sqrt{5}=\frac{a}{b}$$
$$2\sqrt{5}=\frac{a}{b}-3$$
$$\sqrt{5}=\frac{1}{2}\left(\frac{a}{b}-3\right)$$

Since *a* and *b* are integers, $\frac{1}{2}\left(\frac{a}{b}-3\right)$ will also be rational and therefore, $\sqrt{5}$ is rational.

This contradicts the fact that $\sqrt{5}$ is irrational. Hence, our assumption that $3+2\sqrt{5}$ is rational is false. Therefore, $3+2\sqrt{5}$ is irrational.

Question 3:

Prove that the following are irrationals:

(i) $\frac{1}{\sqrt{2}}$ (ii) $7\sqrt{5}$ (iii) $6+\sqrt{2}$

Answer:

(i) $\frac{1}{\sqrt{2}}$

Let $\frac{1}{\sqrt{2}}$ is rational.

Therefore, we can find two integers *a*, *b* (*b* ≠ 0) such that

$$\frac{1}{\sqrt{2}}=\frac{a}{b}$$

$$\sqrt{2}=\frac{b}{a}$$

$\frac{b}{a}$ is rational as *a* and *b* are integers.

Therefore, $\sqrt{2}$ is rational which contradicts to the fact that $\sqrt{2}$ is irrational.

Hence, our assumption is false and $\frac{1}{\sqrt{2}}$ is irrational.

(ii) $7\sqrt{5}$

Let $7\sqrt{5}$ is rational.

Therefore, we can find two integers *a*, *b* (*b* ≠ 0) such that

$7\sqrt{5} = \frac{a}{b}$ for some integers *a* and *b*

$\therefore \sqrt{5} = \frac{a}{7b}$

$\frac{a}{7b}$ is rational as *a* and *b* are integers.

Therefore, $\sqrt{5}$ should be rational.

This contradicts the fact that $\sqrt{5}$ is irrational. Therefore, our assumption that $7\sqrt{5}$ is rational is false. Hence, $7\sqrt{5}$ is irrational.

(iii) $6+\sqrt{2}$

Let $6+\sqrt{2}$ be rational.

Therefore, we can find two integers *a*, *b* (*b* ≠ 0) such that

$$6+\sqrt{2} = \frac{a}{b}$$
$$\sqrt{2} = \frac{a}{b} - 6$$

Since *a* and *b* are integers, $\frac{a}{b} - 6$ is also rational and hence, $\sqrt{2}$ should be rational. This contradicts the fact that $\sqrt{2}$ is irrational. Therefore, our assumption is false and hence, $6+\sqrt{2}$ is irrational.