

Exercise 7.5

Question 1:

$$\frac{x}{(x+1)(x+2)}$$

Answer

$$\text{Let } \frac{x}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$$

$$\Rightarrow x = A(x+2) + B(x+1)$$

Equating the coefficients of x and constant term, we obtain

$$A + B = 1$$

$$2A + B = 0$$

On solving, we obtain

$$A = -1 \text{ and } B = 2$$

$$\therefore \frac{x}{(x+1)(x+2)} = \frac{-1}{(x+1)} + \frac{2}{(x+2)}$$

$$\begin{aligned}\Rightarrow \int \frac{x}{(x+1)(x+2)} dx &= \int \frac{-1}{(x+1)} + \frac{2}{(x+2)} dx \\ &= -\log|x+1| + 2\log|x+2| + C \\ &= \log(x+2)^2 - \log|x+1| + C \\ &= \log \frac{(x+2)^2}{(x+1)} + C\end{aligned}$$

Question 2:

$$\frac{1}{x^2-9}$$

Answer

$$\text{Let } \frac{1}{(x+3)(x-3)} = \frac{A}{(x+3)} + \frac{B}{(x-3)}$$

$$1 = A(x-3) + B(x+3)$$

Equating the coefficients of x and constant term, we obtain

$$A + B = 0$$

$$-3A + 3B = 1$$

On solving, we obtain

$$A = -\frac{1}{6} \text{ and } B = \frac{1}{6}$$

$$\therefore \frac{1}{(x+3)(x-3)} = \frac{-1}{6(x+3)} + \frac{1}{6(x-3)}$$

$$\begin{aligned} \Rightarrow \int \frac{1}{(x^2-9)} dx &= \int \left(\frac{-1}{6(x+3)} + \frac{1}{6(x-3)} \right) dx \\ &= -\frac{1}{6} \log|x+3| + \frac{1}{6} \log|x-3| + C \\ &= \frac{1}{6} \log \left| \frac{(x-3)}{(x+3)} \right| + C \end{aligned}$$

Question 3:

$$\frac{3x-1}{(x-1)(x-2)(x-3)}$$

Answer

$$\text{Let } \frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$$

$$3x-1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \quad \dots(1)$$

Substituting $x = 1, 2,$ and 3 respectively in equation (1), we obtain

$$A = 1, B = -5, \text{ and } C = 4$$

$$\therefore \frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{1}{(x-1)} - \frac{5}{(x-2)} + \frac{4}{(x-3)}$$

$$\begin{aligned} \Rightarrow \int \frac{3x-1}{(x-1)(x-2)(x-3)} dx &= \int \left\{ \frac{1}{(x-1)} - \frac{5}{(x-2)} + \frac{4}{(x-3)} \right\} dx \\ &= \log|x-1| - 5\log|x-2| + 4\log|x-3| + C \end{aligned}$$

Question 4:

$$\frac{x}{(x-1)(x-2)(x-3)}$$

Answer

$$\text{Let } \frac{x}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$$

$$x = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \quad \dots(1)$$

Substituting $x = 1, 2$, and 3 respectively in equation (1), we obtain

$$A = \frac{1}{2}, B = -2, \text{ and } C = \frac{3}{2}$$

$$\therefore \frac{x}{(x-1)(x-2)(x-3)} = \frac{1}{2(x-1)} - \frac{2}{(x-2)} + \frac{3}{2(x-3)}$$

$$\begin{aligned} \Rightarrow \int \frac{x}{(x-1)(x-2)(x-3)} dx &= \int \left\{ \frac{1}{2(x-1)} - \frac{2}{(x-2)} + \frac{3}{2(x-3)} \right\} dx \\ &= \frac{1}{2} \log|x-1| - 2 \log|x-2| + \frac{3}{2} \log|x-3| + C \end{aligned}$$

Question 5:

$$\frac{2x}{x^2+3x+2}$$

Answer

$$\text{Let } \frac{2x}{x^2+3x+2} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$$

$$2x = A(x+2) + B(x+1) \quad \dots(1)$$

Substituting $x = -1$ and -2 in equation (1), we obtain

$$A = -2 \text{ and } B = 4$$

$$\begin{aligned}\therefore \frac{2x}{(x+1)(x+2)} &= \frac{-2}{(x+1)} + \frac{4}{(x+2)} \\ \Rightarrow \int \frac{2x}{(x+1)(x+2)} dx &= \int \left\{ \frac{4}{(x+2)} - \frac{2}{(x+1)} \right\} dx \\ &= 4 \log|x+2| - 2 \log|x+1| + C\end{aligned}$$

Question 6:

$$\frac{1-x^2}{x(1-2x)}$$

Answer

It can be seen that the given integrand is not a proper fraction.

Therefore, on dividing $(1 - x^2)$ by $x(1 - 2x)$, we obtain

$$\frac{1-x^2}{x(1-2x)} = \frac{1}{2} + \frac{1}{2} \left(\frac{2-x}{x(1-2x)} \right)$$

$$\text{Let } \frac{2-x}{x(1-2x)} = \frac{A}{x} + \frac{B}{(1-2x)}$$

$$\Rightarrow (2-x) = A(1-2x) + Bx \quad \dots(1)$$

Substituting $x = 0$ and $\frac{1}{2}$ in equation (1), we obtain

$$A = 2 \text{ and } B = 3$$

$$\therefore \frac{2-x}{x(1-2x)} = \frac{2}{x} + \frac{3}{1-2x}$$

Substituting in equation (1), we obtain

$$\begin{aligned}\frac{1-x^2}{x(1-2x)} &= \frac{1}{2} + \frac{1}{2} \left\{ \frac{2}{x} + \frac{3}{1-2x} \right\} \\ \Rightarrow \int \frac{1-x^2}{x(1-2x)} dx &= \int \left\{ \frac{1}{2} + \frac{1}{2} \left(\frac{2}{x} + \frac{3}{1-2x} \right) \right\} dx \\ &= \frac{x}{2} + \log|x| + \frac{3}{2(-2)} \log|1-2x| + C \\ &= \frac{x}{2} + \log|x| - \frac{3}{4} \log|1-2x| + C\end{aligned}$$

Question 7:

$$\frac{x}{(x^2+1)(x-1)}$$

Answer

$$\text{Let } \frac{x}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1}$$

$$x = (Ax+B)(x-1) + C(x^2+1)$$

$$x = Ax^2 - Ax + Bx - B + Cx^2 + C$$

Equating the coefficients of x^2 , x , and constant term, we obtain

$$A + C = 0$$

$$-A + B = 1$$

$$-B + C = 0$$

On solving these equations, we obtain

$$A = -\frac{1}{2}, B = \frac{1}{2}, \text{ and } C = \frac{1}{2}$$

From equation (1), we obtain

$$\begin{aligned}\therefore \frac{x}{(x^2+1)(x-1)} &= \frac{\left(-\frac{1}{2}x + \frac{1}{2}\right)}{x^2+1} + \frac{\frac{1}{2}}{(x-1)} \\ \Rightarrow \int \frac{x}{(x^2+1)(x-1)} &= -\frac{1}{2} \int \frac{x}{x^2+1} dx + \frac{1}{2} \int \frac{1}{x^2+1} dx + \frac{1}{2} \int \frac{1}{x-1} dx \\ &= -\frac{1}{4} \int \frac{2x}{x^2+1} dx + \frac{1}{2} \tan^{-1} x + \frac{1}{2} \log|x-1| + C\end{aligned}$$

Consider $\int \frac{2x}{x^2+1} dx$, let $(x^2+1) = t \Rightarrow 2x dx = dt$

$$\Rightarrow \int \frac{2x}{x^2+1} dx = \int \frac{dt}{t} = \log|t| = \log|x^2+1|$$

$$\begin{aligned}\therefore \int \frac{x}{(x^2+1)(x-1)} &= -\frac{1}{4} \log|x^2+1| + \frac{1}{2} \tan^{-1} x + \frac{1}{2} \log|x-1| + C \\ &= \frac{1}{2} \log|x-1| - \frac{1}{4} \log|x^2+1| + \frac{1}{2} \tan^{-1} x + C\end{aligned}$$

Question 8:

$$\frac{x}{(x-1)^2(x+2)}$$

Answer

$$\text{Let } \frac{x}{(x-1)^2(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)}$$

$$x = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

Substituting $x = 1$, we obtain

$$B = \frac{1}{3}$$

Equating the coefficients of x^2 and constant term, we obtain

$$A + C = 0$$

$$-2A + 2B + C = 0$$

On solving, we obtain

$$A = \frac{2}{9} \text{ and } C = \frac{-2}{9}$$

$$\therefore \frac{x}{(x-1)^2(x+2)} = \frac{2}{9(x-1)} + \frac{1}{3(x-1)^2} - \frac{2}{9(x+2)}$$

$$\begin{aligned} \Rightarrow \int \frac{x}{(x-1)^2(x+2)} dx &= \frac{2}{9} \int \frac{1}{(x-1)} dx + \frac{1}{3} \int \frac{1}{(x-1)^2} dx - \frac{2}{9} \int \frac{1}{(x+2)} dx \\ &= \frac{2}{9} \log|x-1| + \frac{1}{3} \left(\frac{-1}{x-1} \right) - \frac{2}{9} \log|x+2| + C \\ &= \frac{2}{9} \log \left| \frac{x-1}{x+2} \right| - \frac{1}{3(x-1)} + C \end{aligned}$$

Question 9:

$$\frac{3x+5}{x^3-x^2-x+1}$$

Answer

$$\frac{3x+5}{x^3-x^2-x+1} = \frac{3x+5}{(x-1)^2(x+1)}$$

$$\text{Let } \frac{3x+5}{(x-1)^2(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)}$$

$$3x+5 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

$$3x+5 = A(x^2-1) + B(x+1) + C(x^2+1-2x) \quad \dots(1)$$

Substituting $x = 1$ in equation (1), we obtain

$$B = 4$$

Equating the coefficients of x^2 and x , we obtain

$$A + C = 0$$

$$B - 2C = 3$$

On solving, we obtain

$$A = -\frac{1}{2} \text{ and } C = \frac{1}{2}$$

$$\begin{aligned}
 \therefore \frac{3x+5}{(x-1)^2(x+1)} &= \frac{-1}{2(x-1)} + \frac{4}{(x-1)^2} + \frac{1}{2(x+1)} \\
 \Rightarrow \int \frac{3x+5}{(x-1)^2(x+1)} dx &= -\frac{1}{2} \int \frac{1}{x-1} dx + 4 \int \frac{1}{(x-1)^2} dx + \frac{1}{2} \int \frac{1}{(x+1)} dx \\
 &= -\frac{1}{2} \log|x-1| + 4 \left(\frac{-1}{x-1} \right) + \frac{1}{2} \log|x+1| + C \\
 &= \frac{1}{2} \log \left| \frac{x+1}{x-1} \right| - \frac{4}{(x-1)} + C
 \end{aligned}$$

Question 10:

$$\frac{2x-3}{(x^2-1)(2x+3)}$$

Answer

$$\frac{2x-3}{(x^2-1)(2x+3)} = \frac{2x-3}{(x+1)(x-1)(2x+3)}$$

$$\text{Let } \frac{2x-3}{(x+1)(x-1)(2x+3)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{C}{(2x+3)}$$

$$\Rightarrow (2x-3) = A(x-1)(2x+3) + B(x+1)(2x+3) + C(x+1)(x-1)$$

$$\Rightarrow (2x-3) = A(2x^2+x-3) + B(2x^2+5x+3) + C(x^2-1)$$

$$\Rightarrow (2x-3) = (2A+2B+C)x^2 + (A+5B)x + (-3A+3B-C)$$

Equating the coefficients of x^2 and x , we obtain

$$B = -\frac{1}{10}, A = \frac{5}{2}, \text{ and } C = -\frac{24}{5}$$

$$\begin{aligned}
 \therefore \frac{2x-3}{(x+1)(x-1)(2x+3)} &= \frac{5}{2(x+1)} - \frac{1}{10(x-1)} - \frac{24}{5(2x+3)} \\
 \Rightarrow \int \frac{2x-3}{(x^2-1)(2x+3)} dx &= \frac{5}{2} \int \frac{1}{(x+1)} dx - \frac{1}{10} \int \frac{1}{x-1} dx - \frac{24}{5} \int \frac{1}{(2x+3)} dx \\
 &= \frac{5}{2} \log|x+1| - \frac{1}{10} \log|x-1| - \frac{24}{5 \times 2} \log|2x+3| \\
 &= \frac{5}{2} \log|x+1| - \frac{1}{10} \log|x-1| - \frac{12}{5} \log|2x+3| + C
 \end{aligned}$$

Question 11:

$$\frac{5x}{(x+1)(x^2-4)}$$

Answer

$$\frac{5x}{(x+1)(x^2-4)} = \frac{5x}{(x+1)(x+2)(x-2)}$$

$$\text{Let } \frac{5x}{(x+1)(x+2)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x-2)}$$

$$5x = A(x+2)(x-2) + B(x+1)(x-2) + C(x+1)(x+2) \quad \dots(1)$$

Substituting $x = -1, -2$, and 2 respectively in equation (1), we obtain

$$A = \frac{5}{3}, B = -\frac{5}{2}, \text{ and } C = \frac{5}{6}$$

$$\begin{aligned}
 \therefore \frac{5x}{(x+1)(x+2)(x-2)} &= \frac{5}{3(x+1)} - \frac{5}{2(x+2)} + \frac{5}{6(x-2)} \\
 \Rightarrow \int \frac{5x}{(x+1)(x^2-4)} dx &= \frac{5}{3} \int \frac{1}{(x+1)} dx - \frac{5}{2} \int \frac{1}{(x+2)} dx + \frac{5}{6} \int \frac{1}{(x-2)} dx \\
 &= \frac{5}{3} \log|x+1| - \frac{5}{2} \log|x+2| + \frac{5}{6} \log|x-2| + C
 \end{aligned}$$

Question 12:

$$\frac{x^3+x+1}{x^2-1}$$

Answer

It can be seen that the given integrand is not a proper fraction.

Therefore, on dividing $(x^3 + x + 1)$ by $x^2 - 1$, we obtain

$$\frac{x^3 + x + 1}{x^2 - 1} = x + \frac{2x + 1}{x^2 - 1}$$

$$\text{Let } \frac{2x + 1}{x^2 - 1} = \frac{A}{(x + 1)} + \frac{B}{(x - 1)}$$

$$2x + 1 = A(x - 1) + B(x + 1) \quad \dots(1)$$

Substituting $x = 1$ and -1 in equation (1), we obtain

$$A = \frac{1}{2} \text{ and } B = \frac{3}{2}$$

$$\therefore \frac{x^3 + x + 1}{x^2 - 1} = x + \frac{1}{2(x + 1)} + \frac{3}{2(x - 1)}$$

$$\begin{aligned} \Rightarrow \int \frac{x^3 + x + 1}{x^2 - 1} dx &= \int x dx + \frac{1}{2} \int \frac{1}{(x + 1)} dx + \frac{3}{2} \int \frac{1}{(x - 1)} dx \\ &= \frac{x^2}{2} + \frac{1}{2} \log|x + 1| + \frac{3}{2} \log|x - 1| + C \end{aligned}$$

Question 13:

$$\frac{2}{(1 - x)(1 + x^2)}$$

Answer

$$\text{Let } \frac{2}{(1 - x)(1 + x^2)} = \frac{A}{(1 - x)} + \frac{Bx + C}{(1 + x^2)}$$

$$2 = A(1 + x^2) + (Bx + C)(1 - x)$$

$$2 = A + Ax^2 + Bx - Bx^2 + C - Cx$$

Equating the coefficient of x^2 , x , and constant term, we obtain

$$A - B = 0$$

$$B - C = 0$$

$$A + C = 2$$

On solving these equations, we obtain

$$A = 1, B = 1, \text{ and } C = 1$$

$$\therefore \frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{x+1}{1+x^2}$$

$$\begin{aligned}\Rightarrow \int \frac{2}{(1-x)(1+x^2)} dx &= \int \frac{1}{1-x} dx + \int \frac{x}{1+x^2} dx + \int \frac{1}{1+x^2} dx \\ &= -\int \frac{1}{x-1} dx + \frac{1}{2} \int \frac{2x}{1+x^2} dx + \int \frac{1}{1+x^2} dx \\ &= -\log|x-1| + \frac{1}{2} \log|1+x^2| + \tan^{-1} x + C\end{aligned}$$

Question 14:

$$\frac{3x-1}{(x+2)^2}$$

Answer

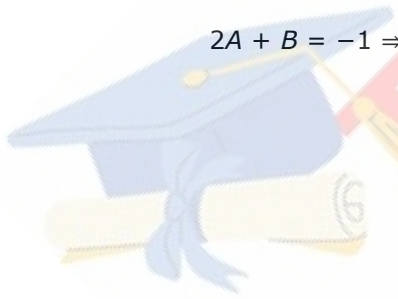
$$\text{Let } \frac{3x-1}{(x+2)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2}$$

$$\Rightarrow 3x-1 = A(x+2) + B$$

Equating the coefficient of x and constant term, we obtain

$$A = 3$$

$$2A + B = -1 \Rightarrow B = -7$$



$$\begin{aligned}
 \therefore \frac{3x-1}{(x+2)^2} &= \frac{3}{(x+2)} - \frac{7}{(x+2)^2} \\
 \Rightarrow \int \frac{3x-1}{(x+2)^2} dx &= 3 \int \frac{1}{(x+2)} dx - 7 \int \frac{x}{(x+2)^2} dx \\
 &= 3 \log|x+2| - 7 \left(\frac{-1}{(x+2)} \right) + C \\
 &= 3 \log|x+2| + \frac{7}{(x+2)} + C
 \end{aligned}$$

Question 15:

$$\frac{1}{x^4-1}$$

Answer

$$\frac{1}{(x^4-1)} = \frac{1}{(x^2-1)(x^2+1)} = \frac{1}{(x+1)(x-1)(1+x^2)}$$

$$\text{Let } \frac{1}{(x+1)(x-1)(1+x^2)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{Cx+D}{(x^2+1)}$$

$$1 = A(x-1)(x^2+1) + B(x+1)(x^2+1) + (Cx+D)(x^2-1)$$

$$1 = A(x^3+x-x^2-1) + B(x^3+x+x^2+1) + Cx^3 + Dx^2 - Cx - D$$

$$1 = (A+B+C)x^3 + (-A+B+D)x^2 + (A+B-C)x + (-A+B-D)$$

Equating the coefficient of x^3 , x^2 , x , and constant term, we obtain

$$A+B+C=0$$

$$-A+B+D=0$$

$$A+B-C=0$$

$$-A+B-D=1$$

On solving these equations, we obtain

$$A = -\frac{1}{4}, B = \frac{1}{4}, C = 0, \text{ and } D = -\frac{1}{2}$$

$$\begin{aligned}\therefore \frac{1}{x^4-1} &= \frac{-1}{4(x+1)} + \frac{1}{4(x-1)} - \frac{1}{2(x^2+1)} \\ \Rightarrow \int \frac{1}{x^4-1} dx &= -\frac{1}{4} \log|x-1| + \frac{1}{4} \log|x+1| - \frac{1}{2} \tan^{-1} x + C \\ &= \frac{1}{4} \log \left| \frac{x-1}{x+1} \right| - \frac{1}{2} \tan^{-1} x + C\end{aligned}$$

Question 16:

$$\frac{1}{x(x^n+1)} \quad [\text{Hint: multiply numerator and denominator by } x^{n-1} \text{ and put } x^n = t]$$

Answer

$$\frac{1}{x(x^n+1)}$$

Multiplying numerator and denominator by x^{n-1} , we obtain

$$\frac{1}{x(x^n+1)} = \frac{x^{n-1}}{x^{n-1}x(x^n+1)} = \frac{x^{n-1}}{x^n(x^n+1)}$$

$$\text{Let } x^n = t \Rightarrow x^{n-1} dx = dt$$

$$\therefore \int \frac{1}{x(x^n+1)} dx = \int \frac{x^{n-1}}{x^n(x^n+1)} dx = \frac{1}{n} \int \frac{1}{t(t+1)} dt$$

$$\text{Let } \frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{(t+1)}$$

$$1 = A(1+t) + Bt \quad \dots(1)$$

Substituting $t = 0, -1$ in equation (1), we obtain

$$A = 1 \text{ and } B = -1$$

$$\therefore \frac{1}{t(t+1)} = \frac{1}{t} - \frac{1}{(1+t)}$$

$$\begin{aligned}
 \Rightarrow \int \frac{1}{x(x^n+1)} dx &= \frac{1}{n} \int \left\{ \frac{1}{t} - \frac{1}{(t+1)} \right\} dx \\
 &= \frac{1}{n} [\log|t| - \log|t+1|] + C \\
 &= -\frac{1}{n} [\log|x^n| - \log|x^n+1|] + C \\
 &= \frac{1}{n} \log \left| \frac{x^n}{x^n+1} \right| + C
 \end{aligned}$$

Question 17:

$$\frac{\cos x}{(1-\sin x)(2-\sin x)} \quad [\text{Hint: Put } \sin x = t]$$

Answer

$$\frac{\cos x}{(1-\sin x)(2-\sin x)}$$

$$\text{Let } \sin x = t \Rightarrow \cos x dx = dt$$

$$\therefore \int \frac{\cos x}{(1-\sin x)(2-\sin x)} dx = \int \frac{dt}{(1-t)(2-t)}$$

$$\text{Let } \frac{1}{(1-t)(2-t)} = \frac{A}{(1-t)} + \frac{B}{(2-t)}$$

$$1 = A(2-t) + B(1-t) \quad \dots(1)$$

Substituting $t = 2$ and then $t = 1$ in equation (1), we obtain

$$A = 1 \text{ and } B = -1$$

$$\therefore \frac{1}{(1-t)(2-t)} = \frac{1}{(1-t)} - \frac{1}{(2-t)}$$

$$\begin{aligned}
 \Rightarrow \int \frac{\cos x}{(1 - \sin x)(2 - \sin x)} dx &= \int \left\{ \frac{1}{1-t} - \frac{1}{(2-t)} \right\} dt \\
 &= -\log|1-t| + \log|2-t| + C \\
 &= \log \left| \frac{2-t}{1-t} \right| + C \\
 &= \log \left| \frac{2 - \sin x}{1 - \sin x} \right| + C
 \end{aligned}$$

Question 18:

$$\frac{(x^2 + 1)(x^2 + 2)}{(x^2 + 3)(x^2 + 4)}$$

Answer

$$\frac{(x^2 + 1)(x^2 + 2)}{(x^2 + 3)(x^2 + 4)} = 1 - \frac{(4x^2 + 10)}{(x^2 + 3)(x^2 + 4)}$$

$$\text{Let } \frac{4x^2 + 10}{(x^2 + 3)(x^2 + 4)} = \frac{Ax + B}{(x^2 + 3)} + \frac{Cx + D}{(x^2 + 4)}$$

$$4x^2 + 10 = (Ax + B)(x^2 + 4) + (Cx + D)(x^2 + 3)$$

$$4x^2 + 10 = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + 3Cx + Dx^2 + 3D$$

$$4x^2 + 10 = (A + C)x^3 + (B + D)x^2 + (4A + 3C)x + (4B + 3D)$$

Equating the coefficients of x^3 , x^2 , x , and constant term, we obtain

$$A + C = 0$$

$$B + D = 4$$

$$4A + 3C = 0$$

$$4B + 3D = 10$$

On solving these equations, we obtain

$$A = 0, B = -2, C = 0, \text{ and } D = 6$$

$$\therefore \frac{4x^2 + 10}{(x^2 + 3)(x^2 + 4)} = \frac{-2}{(x^2 + 3)} + \frac{6}{(x^2 + 4)}$$

$$\begin{aligned}
 \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} &= 1 - \left(\frac{-2}{(x^2+3)} + \frac{6}{(x^2+4)} \right) \\
 \Rightarrow \int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} dx &= \int \left\{ 1 + \frac{2}{(x^2+3)} - \frac{6}{(x^2+4)} \right\} dx \\
 &= \int \left\{ 1 + \frac{2}{x^2 + (\sqrt{3})^2} - \frac{6}{x^2 + 2^2} \right\} dx \\
 &= x + 2 \left(\frac{1}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} \right) - 6 \left(\frac{1}{2} \tan^{-1} \frac{x}{2} \right) + C \\
 &= x + \frac{2}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} - 3 \tan^{-1} \frac{x}{2} + C
 \end{aligned}$$

Question 19:

$$\frac{2x}{(x^2+1)(x^2+3)}$$

Answer

$$\frac{2x}{(x^2+1)(x^2+3)}$$

$$\text{Let } x^2 = t \Rightarrow 2x dx = dt$$

$$\therefore \int \frac{2x}{(x^2+1)(x^2+3)} dx = \int \frac{dt}{(t+1)(t+3)} \quad \dots(1)$$

$$\text{Let } \frac{1}{(t+1)(t+3)} = \frac{A}{(t+1)} + \frac{B}{(t+3)}$$

$$1 = A(t+3) + B(t+1) \quad \dots(1)$$

Substituting $t = -3$ and $t = -1$ in equation (1), we obtain

$$A = \frac{1}{2} \text{ and } B = -\frac{1}{2}$$

$$\therefore \frac{1}{(t+1)(t+3)} = \frac{1}{2(t+1)} - \frac{1}{2(t+3)}$$

$$\begin{aligned} \Rightarrow \int \frac{2x}{(x^2+1)(x^2+3)} dx &= \int \left[\frac{1}{2(t+1)} - \frac{1}{2(t+3)} \right] dt \\ &= \frac{1}{2} \log |(t+1)| - \frac{1}{2} \log |t+3| + C \\ &= \frac{1}{2} \log \left| \frac{t+1}{t+3} \right| + C \\ &= \frac{1}{2} \log \left| \frac{x^2+1}{x^2+3} \right| + C \end{aligned}$$

Question 20:

$$\frac{1}{x(x^4-1)}$$

Answer

$$\frac{1}{x(x^4-1)}$$

Multiplying numerator and denominator by x^3 , we obtain

$$\begin{aligned} \frac{1}{x(x^4-1)} &= \frac{x^3}{x^4(x^4-1)} \\ \therefore \int \frac{1}{x(x^4-1)} dx &= \int \frac{x^3}{x^4(x^4-1)} dx \end{aligned}$$

$$\text{Let } x^4 = t \Rightarrow 4x^3 dx = dt$$

$$\therefore \int \frac{1}{x(x^4-1)} dx = \frac{1}{4} \int \frac{dt}{t(t-1)}$$

$$\text{Let } \frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{(t-1)}$$

$$1 = A(t-1) + Bt \quad \dots(1)$$

Substituting $t = 0$ and 1 in (1), we obtain

$$A = -1 \text{ and } B = 1$$

$$\Rightarrow \frac{1}{t(t+1)} = \frac{-1}{t} + \frac{1}{t-1}$$

$$\begin{aligned} \Rightarrow \int \frac{1}{x(x^4-1)} dx &= \frac{1}{4} \int \left\{ \frac{-1}{t} + \frac{1}{t-1} \right\} dt \\ &= \frac{1}{4} \left[-\log|t| + \log|t-1| \right] + C \\ &= \frac{1}{4} \log \left| \frac{t-1}{t} \right| + C \\ &= \frac{1}{4} \log \left| \frac{x^4-1}{x^4} \right| + C \end{aligned}$$

Question 21:

$$\frac{1}{(e^x-1)} \quad [\text{Hint: Put } e^x = t]$$

Answer

$$\frac{1}{(e^x-1)}$$

$$\text{Let } e^x = t \Rightarrow e^x dx = dt$$

$$\Rightarrow \int \frac{1}{e^x-1} dx = \int \frac{1}{t-1} \times \frac{dt}{t} = \int \frac{1}{t(t-1)} dt$$

$$\text{Let } \frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{t-1}$$

$$1 = A(t-1) + Bt \quad \dots(1)$$

Substituting $t = 1$ and $t = 0$ in equation (1), we obtain

$$A = -1 \text{ and } B = 1$$

$$\therefore \frac{1}{t(t-1)} = \frac{-1}{t} + \frac{1}{t-1}$$

$$\begin{aligned} \Rightarrow \int \frac{1}{t(t-1)} dt &= \log \left| \frac{t-1}{t} \right| + C \\ &= \log \left| \frac{e^x - 1}{e^x} \right| + C \end{aligned}$$

Question 22:

$$\int \frac{x dx}{(x-1)(x-2)} \text{ equals}$$

A. $\log \left| \frac{(x-1)^2}{x-2} \right| + C$

B. $\log \left| \frac{(x-2)^2}{x-1} \right| + C$

C. $\log \left| \left(\frac{x-1}{x-2} \right)^2 \right| + C$

D. $\log |(x-1)(x-2)| + C$

Answer

$$\text{Let } \frac{x}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)}$$

$$x = A(x-2) + B(x-1) \quad \dots(1)$$

Substituting $x = 1$ and 2 in (1), we obtain

$$A = -1 \text{ and } B = 2$$

$$\begin{aligned}
 \therefore \frac{x}{(x-1)(x-2)} &= -\frac{1}{(x-1)} + \frac{2}{(x-2)} \\
 \Rightarrow \int \frac{x}{(x-1)(x-2)} dx &= \int \left\{ \frac{-1}{(x-1)} + \frac{2}{(x-2)} \right\} dx \\
 &= -\log|x-1| + 2\log|x-2| + C \\
 &= \log \left| \frac{(x-2)^2}{x-1} \right| + C
 \end{aligned}$$

Hence, the correct Answer is B.

Question 23:

$$\int \frac{dx}{x(x^2+1)} \text{ equals}$$

- A.** $\log|x| - \frac{1}{2}\log(x^2+1) + C$
- B.** $\log|x| + \frac{1}{2}\log(x^2+1) + C$
- C.** $-\log|x| + \frac{1}{2}\log(x^2+1) + C$
- D.** $\frac{1}{2}\log|x| + \log(x^2+1) + C$

Answer

$$\text{Let } \frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$1 = A(x^2+1) + (Bx+C)x$$

Equating the coefficients of x^2 , x , and constant term, we obtain

$$A + B = 0$$

$$C = 0$$

$$A = 1$$

On solving these equations, we obtain

$$A = 1, B = -1, \text{ and } C = 0$$

$$\therefore \frac{1}{x(x^2+1)} = \frac{1}{x} + \frac{-x}{x^2+1}$$

$$\Rightarrow \int \frac{1}{x(x^2+1)} dx = \int \left\{ \frac{1}{x} - \frac{x}{x^2+1} \right\} dx$$
$$= \log|x| - \frac{1}{2} \log|x^2+1| + C$$

Hence, the correct Answer is A.

