

Exercise 7.4

Question 1:

$$\frac{3x^2}{x^6 + 1}$$

Answer

$$\text{Let } x^3 = t$$

$$\therefore 3x^2 dx = dt$$

$$\begin{aligned}\Rightarrow \int \frac{3x^2}{x^6 + 1} dx &= \int \frac{dt}{t^2 + 1} \\ &= \tan^{-1} t + C \\ &= \tan^{-1}(x^3) + C\end{aligned}$$

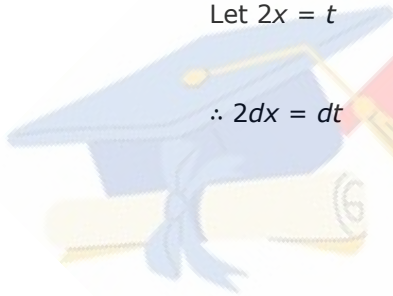
Question 2:

$$\frac{1}{\sqrt{1 + 4x^2}}$$

Answer

$$\text{Let } 2x = t$$

$$\therefore 2dx = dt$$



$$\begin{aligned}
 \Rightarrow \int \frac{1}{\sqrt{1+4x^2}} dx &= \frac{1}{2} \int \frac{dt}{\sqrt{1+t^2}} \\
 &= \frac{1}{2} \left[\log |t + \sqrt{t^2 + 1}| \right] + C & \left[\int \frac{1}{\sqrt{x^2 + a^2}} dt = \log |x + \sqrt{x^2 + a^2}| \right] \\
 &= \frac{1}{2} \log |2x + \sqrt{4x^2 + 1}| + C
 \end{aligned}$$

Question 3:

$$\frac{1}{\sqrt{(2-x)^2 + 1}}$$

Answer

$$\text{Let } 2 - x = t$$

$$\Rightarrow -dx = dt$$

$$\begin{aligned}
 \Rightarrow \int \frac{1}{\sqrt{(2-x)^2 + 1}} dx &= - \int \frac{1}{\sqrt{t^2 + 1}} dt \\
 &= - \log |t + \sqrt{t^2 + 1}| + C & \left[\int \frac{1}{\sqrt{x^2 + a^2}} dt = \log |x + \sqrt{x^2 + a^2}| \right] \\
 &= - \log |2 - x + \sqrt{(2-x)^2 + 1}| + C \\
 &= \log \left| \frac{1}{(2-x) + \sqrt{x^2 - 4x + 5}} \right| + C
 \end{aligned}$$

Question 4:

$$\frac{1}{\sqrt{9-25x^2}}$$

Answer

$$\text{Let } 5x = t$$



$$\therefore 5dx = dt$$

$$\begin{aligned}\Rightarrow \int \frac{1}{\sqrt{9-25x^2}} dx &= \frac{1}{5} \int \frac{1}{9-t^2} dt \\ &= \frac{1}{5} \int \frac{1}{\sqrt{3^2-t^2}} dt \\ &= \frac{1}{5} \sin^{-1} \left(\frac{t}{3} \right) + C \\ &= \frac{1}{5} \sin^{-1} \left(\frac{5x}{3} \right) + C\end{aligned}$$

Question 5:

$$\frac{3x}{1+2x^4}$$

Answer

$$\text{Let } \sqrt{2}x^2 = t$$

$$\therefore 2\sqrt{2}x dx = dt$$

$$\begin{aligned}\Rightarrow \int \frac{3x}{1+2x^4} dx &= \frac{3}{2\sqrt{2}} \int \frac{dt}{1+t^2} \\ &= \frac{3}{2\sqrt{2}} [\tan^{-1} t] + C \\ &= \frac{3}{2\sqrt{2}} \tan^{-1} (\sqrt{2}x^2) + C\end{aligned}$$

Question 6:

$$\frac{x^2}{1-x^6}$$

Answer

$$\text{Let } x^3 = t$$

$$\therefore 3x^2 dx = dt$$

$$\begin{aligned}\Rightarrow \int \frac{x^2}{1-x^6} dx &= \frac{1}{3} \int \frac{dt}{1-t^2} \\ &= \frac{1}{3} \left[\frac{1}{2} \log \left| \frac{1+t}{1-t} \right| \right] + C \\ &= \frac{1}{6} \log \left| \frac{1+x^3}{1-x^3} \right| + C\end{aligned}$$

Question 7:

$$\frac{x-1}{\sqrt{x^2-1}}$$

Answer

$$\int \frac{x-1}{\sqrt{x^2-1}} dx = \int \frac{x}{\sqrt{x^2-1}} dx - \int \frac{1}{\sqrt{x^2-1}} dx \quad \dots(1)$$

For $\int \frac{x}{\sqrt{x^2-1}} dx$, let $x^2-1=t \Rightarrow 2x dx = dt$

$$\begin{aligned}\therefore \int \frac{x}{\sqrt{x^2-1}} dx &= \frac{1}{2} \int \frac{dt}{\sqrt{t}} \\ &= \frac{1}{2} \int t^{-\frac{1}{2}} dt \\ &= \frac{1}{2} \left[2t^{\frac{1}{2}} \right] \\ &= \sqrt{t} \\ &= \sqrt{x^2-1}\end{aligned}$$

From (1), we obtain



$$\begin{aligned}\int \frac{x-1}{\sqrt{x^2-1}} dx &= \int \frac{x}{\sqrt{x^2-1}} dx - \int \frac{1}{\sqrt{x^2-1}} dx \\ &= \sqrt{x^2-1} - \log|x + \sqrt{x^2-1}| + C\end{aligned}$$

$$\left[\int \frac{1}{\sqrt{x^2-a^2}} dt = \log|x + \sqrt{x^2-a^2}| \right]$$

Question 8:

$$\frac{x^2}{\sqrt{x^6+a^6}}$$

Answer

Let $x^3 = t$

$$\Rightarrow 3x^2 dx = dt$$

$$\begin{aligned}\therefore \int \frac{x^2}{\sqrt{x^6+a^6}} dx &= \frac{1}{3} \int \frac{dt}{\sqrt{t^2+(a^3)^2}} \\ &= \frac{1}{3} \log|t + \sqrt{t^2+a^6}| + C \\ &= \frac{1}{3} \log|x^3 + \sqrt{x^6+a^6}| + C\end{aligned}$$

Question 9:

$$\frac{\sec^2 x}{\sqrt{\tan^2 x + 4}}$$

Answer

Let $\tan x = t$

$$\therefore \sec^2 x dx = dt$$

$$\begin{aligned}
 \Rightarrow \int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}} dx &= \int \frac{dt}{\sqrt{t^2 + 2^2}} \\
 &= \log \left| t + \sqrt{t^2 + 4} \right| + C \\
 &= \log \left| \tan x + \sqrt{\tan^2 x + 4} \right| + C
 \end{aligned}$$

Question 10:

$$\frac{1}{\sqrt{x^2 + 2x + 2}}$$

Answer

$$\int \frac{1}{\sqrt{x^2 + 2x + 2}} dx = \int \frac{1}{\sqrt{(x+1)^2 + (1)^2}} dx$$

Let $x+1 = t$

$\therefore dx = dt$

$$\begin{aligned}
 \Rightarrow \int \frac{1}{\sqrt{x^2 + 2x + 2}} dx &= \int \frac{1}{\sqrt{t^2 + 1}} dt \\
 &= \log \left| t + \sqrt{t^2 + 1} \right| + C \\
 &= \log \left| (x+1) + \sqrt{(x+1)^2 + 1} \right| + C \\
 &= \log \left| (x+1) + \sqrt{x^2 + 2x + 2} \right| + C
 \end{aligned}$$

Question 11:

$$\frac{1}{\sqrt{9x^2 + 6x + 5}}$$

Answer

$$\int \frac{1}{9x^2 + 6x + 5} dx = \int \frac{1}{(3x+1)^2 + (2)^2} dx$$

$$\text{Let } (3x+1) = t$$

$$\therefore 3dx = dt$$

$$\begin{aligned}\Rightarrow \int \frac{1}{(3x+1)^2 + (2)^2} dx &= \frac{1}{3} \int \frac{1}{t^2 + 2^2} dt \\ &= \frac{1}{3} \left[\frac{1}{2} \tan^{-1} \left(\frac{t}{2} \right) \right] + C \\ &= \frac{1}{6} \tan^{-1} \left(\frac{3x+1}{2} \right) + C\end{aligned}$$

Question 12:

$$\frac{1}{\sqrt{7-6x-x^2}}$$

Answer

$7-6x-x^2$ can be written as $7-(x^2+6x+9-9)$.

Therefore,

$$7-(x^2+6x+9-9)$$

$$= 16-(x^2+6x+9)$$

$$= 16-(x+3)^2$$

$$= (4)^2 - (x+3)^2$$

$$\therefore \int \frac{1}{\sqrt{7-6x-x^2}} dx = \int \frac{1}{\sqrt{(4)^2 - (x+3)^2}} dx$$

$$\text{Let } x+3 = t$$

$$\Rightarrow dx = dt$$

$$\begin{aligned}\Rightarrow \int \frac{1}{\sqrt{(4)^2 - (x+3)^2}} dx &= \int \frac{1}{\sqrt{(4)^2 - (t)^2}} dt \\ &= \sin^{-1} \left(\frac{t}{4} \right) + C \\ &= \sin^{-1} \left(\frac{x+3}{4} \right) + C\end{aligned}$$

Question 13:

$$\frac{1}{\sqrt{(x-1)(x-2)}}$$

Answer

$(x-1)(x-2)$ can be written as $x^2 - 3x + 2$.

Therefore,

$$x^2 - 3x + 2$$

$$= x^2 - 3x + \frac{9}{4} - \frac{9}{4} + 2$$

$$= \left(x - \frac{3}{2}\right)^2 - \frac{1}{4}$$

$$= \left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2$$

$$\therefore \int \frac{1}{\sqrt{(x-1)(x-2)}} dx = \int \frac{1}{\sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} dx$$

$$\text{Let } x - \frac{3}{2} = t$$

$$\therefore dx = dt$$

$$\Rightarrow \int \frac{1}{\sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} dx = \int \frac{1}{\sqrt{t^2 - \left(\frac{1}{2}\right)^2}} dt$$

$$= \log \left| t + \sqrt{t^2 - \left(\frac{1}{2}\right)^2} \right| + C$$

$$= \log \left| \left(x - \frac{3}{2}\right) + \sqrt{x^2 - 3x + 2} \right| + C$$

Question 14:

$$\frac{1}{\sqrt{8+3x-x^2}}$$

Answer

$8+3x-x^2$ can be written as $8-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)$.

Therefore,

$$8-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)$$

$$=\frac{41}{4}-\left(x-\frac{3}{2}\right)^2$$

$$\Rightarrow \int \frac{1}{\sqrt{8+3x-x^2}} dx = \int \frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}} dx$$

$$\text{Let } x-\frac{3}{2}=t$$

$$\therefore dx=dt$$

$$\Rightarrow \int \frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}} dx = \int \frac{1}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2-t^2}} dt$$

$$= \sin^{-1} \left(\frac{t}{\frac{\sqrt{41}}{2}} \right) + C$$

$$= \sin^{-1} \left(\frac{x-\frac{3}{2}}{\frac{\sqrt{41}}{2}} \right) + C$$

$$= \sin^{-1} \left(\frac{2x-3}{\sqrt{41}} \right) + C$$

Question 15:

$$\frac{1}{\sqrt{(x-a)(x-b)}}$$

Answer



$(x-a)(x-b)$ can be written as $x^2 - (a+b)x + ab$.

Therefore,

$$x^2 - (a+b)x + ab$$

$$= x^2 - (a+b)x + \frac{(a+b)^2}{4} - \frac{(a+b)^2}{4} + ab$$

$$= \left[x - \left(\frac{a+b}{2} \right) \right]^2 - \frac{(a-b)^2}{4}$$

$$\Rightarrow \int \frac{1}{\sqrt{(x-a)(x-b)}} dx = \int \frac{1}{\sqrt{\left\{ x - \left(\frac{a+b}{2} \right) \right\}^2 - \left(\frac{a-b}{2} \right)^2}} dx$$

$$\text{Let } x - \left(\frac{a+b}{2} \right) = t$$

$$\therefore dx = dt$$

$$\Rightarrow \int \frac{1}{\sqrt{\left\{ x - \left(\frac{a+b}{2} \right) \right\}^2 - \left(\frac{a-b}{2} \right)^2}} dx = \int \frac{1}{\sqrt{t^2 - \left(\frac{a-b}{2} \right)^2}} dt$$

$$= \log \left| t + \sqrt{t^2 - \left(\frac{a-b}{2} \right)^2} \right| + C$$

$$= \log \left| \left\{ x - \left(\frac{a+b}{2} \right) \right\} + \sqrt{(x-a)(x-b)} \right| + C$$

Question 16:

$$\frac{4x+1}{\sqrt{2x^2+x-3}}$$

Answer

$$\text{Let } 4x+1 = A \frac{d}{dx} (2x^2+x-3) + B$$

$$\Rightarrow 4x+1 = A(4x+1) + B$$

$$\Rightarrow 4x+1 = 4Ax + A + B$$

Equating the coefficients of x and constant term on both sides, we obtain

$$4A = 4 \Rightarrow A = 1$$

$$A + B = 1 \Rightarrow B = 0$$

$$\text{Let } 2x^2 + x - 3 = t$$

$$\therefore (4x + 1) dx = dt$$

$$\begin{aligned}\Rightarrow \int \frac{4x+1}{\sqrt{2x^2+x-3}} dx &= \int \frac{1}{\sqrt{t}} dt \\ &= 2\sqrt{t} + C \\ &= 2\sqrt{2x^2+x-3} + C\end{aligned}$$

Question 17:

$$\frac{x+2}{\sqrt{x^2-1}}$$

Answer

$$\text{Let } x+2 = A \frac{d}{dx}(x^2-1) + B \quad \dots(1)$$

$$\Rightarrow x+2 = A(2x) + B$$

Equating the coefficients of x and constant term on both sides, we obtain

$$2A = 1 \Rightarrow A = \frac{1}{2}$$

$$B = 2$$

From (1), we obtain

$$(x+2) = \frac{1}{2}(2x) + 2$$

$$\begin{aligned} \text{Then, } \int \frac{x+2}{\sqrt{x^2-1}} dx &= \int \frac{\frac{1}{2}(2x) + 2}{\sqrt{x^2-1}} dx \\ &= \frac{1}{2} \int \frac{2x}{\sqrt{x^2-1}} dx + \int \frac{2}{\sqrt{x^2-1}} dx \quad \dots(2) \end{aligned}$$

$$\ln \frac{1}{2} \int \frac{2x}{\sqrt{x^2-1}} dx, \text{ let } x^2 - 1 = t \Rightarrow 2x dx = dt$$

$$\begin{aligned} \frac{1}{2} \int \frac{2x}{\sqrt{x^2-1}} dx &= \frac{1}{2} \int \frac{dt}{\sqrt{t}} \\ &= \frac{1}{2} [2\sqrt{t}] \\ &= \sqrt{t} \\ &= \sqrt{x^2-1} \end{aligned}$$

$$\text{Then, } \int \frac{2}{\sqrt{x^2-1}} dx = 2 \int \frac{1}{\sqrt{x^2-1}} dx = 2 \log |x + \sqrt{x^2-1}|$$

From equation (2), we obtain

$$\int \frac{x+2}{\sqrt{x^2-1}} dx = \sqrt{x^2-1} + 2 \log |x + \sqrt{x^2-1}| + C$$

Question 18:

$$\frac{5x-2}{1+2x+3x^2}$$

Answer

$$\text{Let } 5x-2 = A \frac{d}{dx} (1+2x+3x^2) + B$$

$$\Rightarrow 5x-2 = A(2+6x) + B$$

Equating the coefficient of x and constant term on both sides, we obtain

$$5 = 6A \Rightarrow A = \frac{5}{6}$$

$$2A + B = -2 \Rightarrow B = -\frac{11}{3}$$

$$\therefore 5x - 2 = \frac{5}{6}(2 + 6x) + \left(-\frac{11}{3}\right)$$

$$\begin{aligned}\Rightarrow \int \frac{5x-2}{1+2x+3x^2} dx &= \int \frac{\frac{5}{6}(2+6x) - \frac{11}{3}}{1+2x+3x^2} dx \\ &= \frac{5}{6} \int \frac{2+6x}{1+2x+3x^2} dx - \frac{11}{3} \int \frac{1}{1+2x+3x^2} dx\end{aligned}$$

$$\text{Let } I_1 = \int \frac{2+6x}{1+2x+3x^2} dx \text{ and } I_2 = \int \frac{1}{1+2x+3x^2} dx$$

$$\therefore \int \frac{5x-2}{1+2x+3x^2} dx = \frac{5}{6} I_1 - \frac{11}{3} I_2 \quad \dots(1)$$

$$I_1 = \int \frac{2+6x}{1+2x+3x^2} dx$$

$$\text{Let } 1+2x+3x^2 = t$$

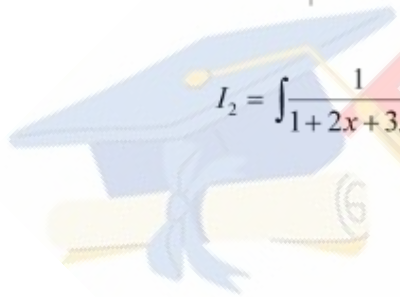
$$\Rightarrow (2+6x) dx = dt$$

$$\therefore I_1 = \int \frac{dt}{t}$$

$$I_1 = \log|t|$$

$$I_1 = \log|1+2x+3x^2| \quad \dots(2)$$

$$I_2 = \int \frac{1}{1+2x+3x^2} dx$$



$1+2x+3x^2$ can be written as $1+3\left(x^2+\frac{2}{3}x\right)$.

Therefore,

$$\begin{aligned} & 1+3\left(x^2+\frac{2}{3}x\right) \\ &= 1+3\left(x^2+\frac{2}{3}x+\frac{1}{9}-\frac{1}{9}\right) \\ &= 1+3\left(x+\frac{1}{3}\right)^2-\frac{1}{3} \\ &= \frac{2}{3}+3\left(x+\frac{1}{3}\right)^2 \\ &= 3\left[\left(x+\frac{1}{3}\right)^2+\frac{2}{9}\right] \\ &= 3\left[\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2\right] \end{aligned}$$

$$\begin{aligned} I_2 &= \frac{1}{3} \int \frac{1}{\left[\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2\right]} dx \\ &= \frac{1}{3} \left[\frac{1}{\frac{\sqrt{2}}{3}} \tan^{-1} \left(\frac{x+\frac{1}{3}}{\frac{\sqrt{2}}{3}} \right) \right] \\ &= \frac{1}{3} \left[\frac{3}{\sqrt{2}} \tan^{-1} \left(\frac{3x+1}{\sqrt{2}} \right) \right] \\ &= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{3x+1}{\sqrt{2}} \right) \quad \dots(3) \end{aligned}$$

Substituting equations (2) and (3) in equation (1), we obtain

$$\begin{aligned}\int \frac{5x-2}{1+2x+3x^2} dx &= \frac{5}{6} \left[\log |1+2x+3x^2| \right] - \frac{11}{3} \left[\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{3x+1}{\sqrt{2}} \right) \right] + C \\ &= \frac{5}{6} \log |1+2x+3x^2| - \frac{11}{3\sqrt{2}} \tan^{-1} \left(\frac{3x+1}{\sqrt{2}} \right) + C\end{aligned}$$

Question 19:

$$\frac{6x+7}{\sqrt{(x-5)(x-4)}}$$

Answer

$$\frac{6x+7}{\sqrt{(x-5)(x-4)}} = \frac{6x+7}{\sqrt{x^2-9x+20}}$$

$$\text{Let } 6x+7 = A \frac{d}{dx} (x^2-9x+20) + B$$

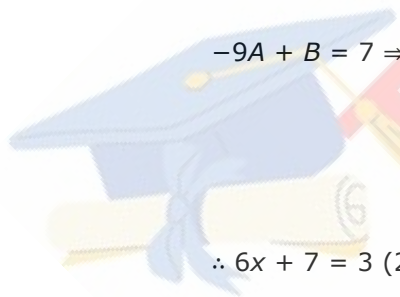
$$\Rightarrow 6x+7 = A(2x-9) + B$$

Equating the coefficients of x and constant term, we obtain

$$2A = 6 \Rightarrow A = 3$$

$$-9A + B = 7 \Rightarrow B = 34$$

$$\therefore 6x+7 = 3(2x-9) + 34$$



$$\begin{aligned}\int \frac{6x+7}{\sqrt{x^2-9x+20}} &= \int \frac{3(2x-9)+34}{\sqrt{x^2-9x+20}} dx \\ &= 3 \int \frac{2x-9}{\sqrt{x^2-9x+20}} dx + 34 \int \frac{1}{\sqrt{x^2-9x+20}} dx\end{aligned}$$

$$\text{Let } I_1 = \int \frac{2x-9}{\sqrt{x^2-9x+20}} dx \text{ and } I_2 = \int \frac{1}{\sqrt{x^2-9x+20}} dx$$

$$\therefore \int \frac{6x+7}{\sqrt{x^2-9x+20}} = 3I_1 + 34I_2 \quad \dots(1)$$

Then,

$$I_1 = \int \frac{2x-9}{\sqrt{x^2-9x+20}} dx$$

$$\text{Let } x^2-9x+20 = t$$

$$\Rightarrow (2x-9)dx = dt$$

$$\Rightarrow I_1 = \int \frac{dt}{\sqrt{t}}$$

$$I_1 = 2\sqrt{t}$$

$$I_1 = 2\sqrt{x^2-9x+20} \quad \dots(2)$$

$$\text{and } I_2 = \int \frac{1}{\sqrt{x^2-9x+20}} dx$$



$x^2 - 9x + 20$ can be written as $x^2 - 9x + 20 + \frac{81}{4} - \frac{81}{4}$.

Therefore,

$$x^2 - 9x + 20 + \frac{81}{4} - \frac{81}{4}$$

$$= \left(x - \frac{9}{2}\right)^2 - \frac{1}{4}$$

$$= \left(x - \frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2$$

$$\Rightarrow I_2 = \int \frac{1}{\sqrt{\left(x - \frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} dx$$

$$I_2 = \log \left| \left(x - \frac{9}{2}\right) + \sqrt{x^2 - 9x + 20} \right| \quad \dots(3)$$

Substituting equations (2) and (3) in (1), we obtain

$$\begin{aligned} \int \frac{6x+7}{\sqrt{x^2-9x+20}} dx &= 3 \left[2\sqrt{x^2-9x+20} \right] + 34 \log \left[\left(x - \frac{9}{2}\right) + \sqrt{x^2-9x+20} \right] + C \\ &= 6\sqrt{x^2-9x+20} + 34 \log \left[\left(x - \frac{9}{2}\right) + \sqrt{x^2-9x+20} \right] + C \end{aligned}$$

Question 20:

$$\frac{x+2}{\sqrt{4x-x^2}}$$

Answer

$$\text{Let } x+2 = A \frac{d}{dx}(4x-x^2) + B$$

$$\Rightarrow x+2 = A(4-2x) + B$$

Equating the coefficients of x and constant term on both sides, we obtain

$$-2A = 1 \Rightarrow A = -\frac{1}{2}$$

$$4A + B = 2 \Rightarrow B = 4$$

$$\Rightarrow (x+2) = -\frac{1}{2}(4-2x) + 4$$

$$\begin{aligned} \therefore \int \frac{x+2}{\sqrt{4x-x^2}} dx &= \int \frac{-\frac{1}{2}(4-2x) + 4}{\sqrt{4x-x^2}} dx \\ &= -\frac{1}{2} \int \frac{4-2x}{\sqrt{4x-x^2}} dx + 4 \int \frac{1}{\sqrt{4x-x^2}} dx \end{aligned}$$

$$\text{Let } I_1 = \int \frac{4-2x}{\sqrt{4x-x^2}} dx \text{ and } I_2 = \int \frac{1}{\sqrt{4x-x^2}} dx$$

$$\therefore \int \frac{x+2}{\sqrt{4x-x^2}} dx = -\frac{1}{2} I_1 + 4 I_2 \quad \dots(1)$$

$$\text{Then, } I_1 = \int \frac{4-2x}{\sqrt{4x-x^2}} dx$$

$$\text{Let } 4x-x^2 = t$$

$$\Rightarrow (4-2x) dx = dt$$

$$\Rightarrow I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{4x-x^2} \quad \dots(2)$$

$$I_2 = \int \frac{1}{\sqrt{4x-x^2}} dx$$

$$\begin{aligned} \Rightarrow 4x-x^2 &= -(-4x+x^2) \\ &= (-4x+x^2+4-4) \\ &= 4-(x-2)^2 \\ &= (2)^2-(x-2)^2 \end{aligned}$$

$$\therefore I_2 = \int \frac{1}{\sqrt{(2)^2-(x-2)^2}} dx = \sin^{-1} \left(\frac{x-2}{2} \right) \quad \dots(3)$$

Using equations (2) and (3) in (1), we obtain

$$\begin{aligned}\int \frac{x+2}{\sqrt{4x-x^2}} dx &= -\frac{1}{2} \left(2\sqrt{4x-x^2} \right) + 4 \sin^{-1} \left(\frac{x-2}{2} \right) + C \\ &= -\sqrt{4x-x^2} + 4 \sin^{-1} \left(\frac{x-2}{2} \right) + C\end{aligned}$$

Question 21:

$$\frac{x+2}{\sqrt{x^2+2x+3}}$$

Answer

$$\begin{aligned}\int \frac{(x+2)}{\sqrt{x^2+2x+3}} dx &= \frac{1}{2} \int \frac{2(x+2)}{\sqrt{x^2+2x+3}} dx \\ &= \frac{1}{2} \int \frac{2x+4}{\sqrt{x^2+2x+3}} dx \\ &= \frac{1}{2} \int \frac{2x+2}{\sqrt{x^2+2x+3}} dx + \frac{1}{2} \int \frac{2}{\sqrt{x^2+2x+3}} dx \\ &= \frac{1}{2} \int \frac{2x+2}{\sqrt{x^2+2x+3}} dx + \int \frac{1}{\sqrt{x^2+2x+3}} dx\end{aligned}$$

$$\text{Let } I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}} dx \text{ and } I_2 = \int \frac{1}{\sqrt{x^2+2x+3}} dx$$

$$\therefore \int \frac{x+2}{\sqrt{x^2+2x+3}} dx = \frac{1}{2} I_1 + I_2 \quad \dots(1)$$

$$\text{Then, } I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}} dx$$

$$\text{Let } x^2 + 2x + 3 = t$$

$$\Rightarrow (2x + 2) dx = dt$$

$$I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{x^2 + 2x + 3} \quad \dots(2)$$

$$I_2 = \int \frac{1}{\sqrt{x^2 + 2x + 3}} dx$$

$$\Rightarrow x^2 + 2x + 3 = x^2 + 2x + 1 + 2 = (x+1)^2 + (\sqrt{2})^2$$

$$\therefore I_2 = \int \frac{1}{\sqrt{(x+1)^2 + (\sqrt{2})^2}} dx = \log \left| (x+1) + \sqrt{x^2 + 2x + 3} \right| \quad \dots(3)$$

Using equations (2) and (3) in (1), we obtain

$$\begin{aligned} \int \frac{x+2}{\sqrt{x^2 + 2x + 3}} dx &= \frac{1}{2} \left[2\sqrt{x^2 + 2x + 3} \right] + \log \left| (x+1) + \sqrt{x^2 + 2x + 3} \right| + C \\ &= \sqrt{x^2 + 2x + 3} + \log \left| (x+1) + \sqrt{x^2 + 2x + 3} \right| + C \end{aligned}$$

Question 22:

$$\frac{x+3}{x^2 - 2x - 5}$$

Answer

$$\text{Let } (x+3) = A \frac{d}{dx} (x^2 - 2x - 5) + B$$

$$(x+3) = A(2x-2) + B$$

Equating the coefficients of x and constant term on both sides, we obtain

$$2A = 1 \Rightarrow A = \frac{1}{2}$$

$$-2A + B = 3 \Rightarrow B = 4$$

$$\therefore (x+3) = \frac{1}{2}(2x-2) + 4$$

$$\begin{aligned} \Rightarrow \int \frac{x+3}{x^2 - 2x - 5} dx &= \int \frac{\frac{1}{2}(2x-2) + 4}{x^2 - 2x - 5} dx \\ &= \frac{1}{2} \int \frac{2x-2}{x^2 - 2x - 5} dx + 4 \int \frac{1}{x^2 - 2x - 5} dx \end{aligned}$$

$$\text{Let } I_1 = \int \frac{2x-2}{x^2-2x-5} dx \text{ and } I_2 = \int \frac{1}{x^2-2x-5} dx$$

$$\therefore \int \frac{x+3}{(x^2-2x-5)} dx = \frac{1}{2} I_1 + 4 I_2 \quad \dots(1)$$

$$\text{Then, } I_1 = \int \frac{2x-2}{x^2-2x-5} dx$$

$$\text{Let } x^2 - 2x - 5 = t$$

$$\Rightarrow (2x-2) dx = dt$$

$$\Rightarrow I_1 = \int \frac{dt}{t} = \log|t| = \log|x^2 - 2x - 5| \quad \dots(2)$$

$$\begin{aligned} I_2 &= \int \frac{1}{x^2-2x-5} dx \\ &= \int \frac{1}{(x^2-2x+1)-6} dx \\ &= \int \frac{1}{(x-1)^2 + (\sqrt{6})^2} dx \\ &= \frac{1}{2\sqrt{6}} \log \left(\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}} \right) \quad \dots(3) \end{aligned}$$

Substituting (2) and (3) in (1), we obtain

$$\begin{aligned} \int \frac{x+3}{x^2-2x-5} dx &= \frac{1}{2} \log|x^2-2x-5| + \frac{4}{2\sqrt{6}} \log \left| \frac{x-1-\sqrt{6}}{x-1+\sqrt{6}} \right| + C \\ &= \frac{1}{2} \log|x^2-2x-5| + \frac{2}{\sqrt{6}} \log \left| \frac{x-1-\sqrt{6}}{x-1+\sqrt{6}} \right| + C \end{aligned}$$

Question 23:

$$\frac{5x+3}{\sqrt{x^2+4x+10}}$$

Answer

$$\text{Let } 5x+3 = A \frac{d}{dx}(x^2+4x+10) + B$$

$$\Rightarrow 5x+3 = A(2x+4) + B$$

Equating the coefficients of x and constant term, we obtain

$$2A = 5 \Rightarrow A = \frac{5}{2}$$

$$4A + B = 3 \Rightarrow B = -7$$

$$\therefore 5x+3 = \frac{5}{2}(2x+4) - 7$$

$$\begin{aligned} \Rightarrow \int \frac{5x+3}{\sqrt{x^2+4x+10}} dx &= \int \frac{\frac{5}{2}(2x+4) - 7}{\sqrt{x^2+4x+10}} dx \\ &= \frac{5}{2} \int \frac{2x+4}{\sqrt{x^2+4x+10}} dx - 7 \int \frac{1}{\sqrt{x^2+4x+10}} dx \end{aligned}$$

$$\text{Let } I_1 = \int \frac{2x+4}{\sqrt{x^2+4x+10}} dx \text{ and } I_2 = \int \frac{1}{\sqrt{x^2+4x+10}} dx$$

$$\therefore \int \frac{5x+3}{\sqrt{x^2+4x+10}} dx = \frac{5}{2} I_1 - 7 I_2 \quad \dots(1)$$

$$\text{Then, } I_1 = \int \frac{2x+4}{\sqrt{x^2+4x+10}} dx$$

$$\text{Let } x^2+4x+10 = t$$

$$\therefore (2x+4)dx = dt$$

$$\Rightarrow I_1 = \int \frac{dt}{t} = 2\sqrt{t} = 2\sqrt{x^2+4x+10} \quad \dots(2)$$

$$\begin{aligned} I_2 &= \int \frac{1}{\sqrt{x^2+4x+10}} dx \\ &= \int \frac{1}{\sqrt{(x^2+4x+4)+6}} dx \\ &= \int \frac{1}{(x+2)^2 + (\sqrt{6})^2} dx \end{aligned}$$

$$= \log \left| (x+2)\sqrt{x^2+4x+10} \right| \quad \dots(3)$$

Using equations (2) and (3) in (1), we obtain

$$\begin{aligned}\int \frac{5x+3}{\sqrt{x^2+4x+10}} dx &= \frac{5}{2} \left[2\sqrt{x^2+4x+10} \right] - 7 \log \left| (x+2) + \sqrt{x^2+4x+10} \right| + C \\ &= 5\sqrt{x^2+4x+10} - 7 \log \left| (x+2) + \sqrt{x^2+4x+10} \right| + C\end{aligned}$$

Question 24:

$$\int \frac{dx}{x^2+2x+2} \text{ equals}$$

- A.** $x \tan^{-1}(x+1) + C$
- B.** $\tan^{-1}(x+1) + C$
- C.** $(x+1) \tan^{-1} x + C$
- D.** $\tan^{-1} x + C$

Answer

$$\begin{aligned}\int \frac{dx}{x^2+2x+2} &= \int \frac{dx}{(x^2+2x+1)+1} \\ &= \int \frac{1}{(x+1)^2+(1)^2} dx \\ &= \left[\tan^{-1}(x+1) \right] + C\end{aligned}$$

Hence, the correct Answer is B.

Question 25:

$$\int \frac{dx}{\sqrt{9x-4x^2}} \text{ equals}$$

- A.** $\frac{1}{9} \sin^{-1} \left(\frac{9x-8}{8} \right) + C$
- B.** $\frac{1}{2} \sin^{-1} \left(\frac{8x-9}{9} \right) + C$
- C.** $\frac{1}{3} \sin^{-1} \left(\frac{9x-8}{8} \right) + C$

D. $\frac{1}{2} \sin^{-1} \left(\frac{9x-8}{9} \right) + C$

Answer

$$\begin{aligned} & \int \frac{dx}{\sqrt{9x-4x^2}} \\ &= \int \frac{1}{\sqrt{-4 \left(x^2 - \frac{9}{4}x \right)}} dx \\ &= \int \frac{1}{-4 \left(x^2 - \frac{9}{4}x + \frac{81}{64} - \frac{81}{64} \right)} dx \\ &= \int \frac{1}{\sqrt{-4 \left[\left(x - \frac{9}{8} \right)^2 - \left(\frac{9}{8} \right)^2 \right]}} dx \\ &= \frac{1}{2} \int \frac{1}{\sqrt{\left(\frac{9}{8} \right)^2 - \left(x - \frac{9}{8} \right)^2}} dx \\ &= \frac{1}{2} \left[\sin^{-1} \left(\frac{x - \frac{9}{8}}{\frac{9}{8}} \right) \right] + C \quad \left(\int \frac{dy}{\sqrt{a^2 - y^2}} = \sin^{-1} \frac{y}{a} + C \right) \\ &= \frac{1}{2} \sin^{-1} \left(\frac{8x-9}{9} \right) + C \end{aligned}$$

Hence, the correct Answer is B.

