Exercise 6.3

Question 1:

Find the maximum and minimum values, if any, of the following functions given by

(i) $f(x) = (2x - 1)^2 + 3$ (ii) $f(x) = 9x^2 + 12x + 2$

(iii) $f(x) = -(x - 1)^2 + 10$ (iv) $g(x) = x^3 + 1$

Answer

(i) The given function is $f(x) = (2x - 1)^2 + 3$.

It can be observed that $(2x - 1)^2 \geq 0$ for every x □ **R**.

Therefore, $f(x) = (2x - 1)^2 + 3 \geq 3$ for every x □ **R**.

The minimum value of f is attained when $2x - 1 = 0$.

$2x - 1 = 0$ □ $x = \frac{1}{2}$

□Minimum value of f = $f\left(\frac{1}{2}\right) = \left(2 \cdot \frac{1}{2} - 1\right)^2 + 3$ = 3

Hence, function f does not have a maximum value.

(ii) The given function is $f(x) = 9x^2 + 12x + 2 = (3x + 2)^2 - 2$.

It can be observed that $(3x + 2)^2 \geq 0$ for every x □ **R**.

Therefore, $f(x) = (3x + 2)^2 - 2 \geq -2$ for every x □ **R**.

The minimum value of f is attained when $3x + 2 = 0$.

$3x + 2 = 0$ □ $x = \frac{-2}{3}$

□Minimum value of f = $f\left(-\frac{2}{3}\right) = \left(3\left(\frac{-2}{3}\right) + 2\right)^2 - 2 = -2$

Hence, function f does not have a maximum value.

(iii) The given function is $f(x) = -(x - 1)^2 + 10$.

It can be observed that $(x - 1)^2 \geq 0$ for every x □ **R**.

Therefore, $f(x) = -(x - 1)^2 + 10 \leq 10$ for every x □ **R**.

The maximum value of f is attained when $(x - 1) = 0$.

$(x - 1) = 0$ □ $x = 0$

□Maximum value of $f = f(1) = -(1 - 1)^2 + 10 = 10$

Hence, function f does not have a minimum value.

(iv) The given function is $g(x) = x^3 + 1$.

Hence, function g neither has a maximum value nor a minimum value.

Question 2:

Find the maximum and minimum values, if any, of the following functions given by

(i) $f(x) = |x + 2| - 1$ (ii) $g(x) = -|x + 1| + 3$

(iii) $h(x) = \sin(2x) + 5$ (iv) $f(x) = |\sin 4x + 3|$

(v) $h(x) = x + 4, x \in (-1, 1)$

Answer

(i) $f(x) = |x+2|-1$

We know that $|x+2| \geq 0$ for every x ☐ **R**.

Therefore, $f(x) = |x+2|-1 \geq -1$ for every x ☐ **R**.

The minimum value of f is attained when $|x+2| = 0$.

$|x+2| = 0$

$\Rightarrow x = -2$

☐Minimum value of $f = f(-2) = |-2+2|-1 = -1$

Hence, function f does not have a maximum value.

(ii) $g(x) = -|x+1|+3$

We know that $-|x+1| \leq 0$ for every x ☐ **R**.

Therefore, $g(x) = -|x+1|+3 \leq 3$ for every x ☐ **R**.

The maximum value of g is attained when $|x+1| = 0$.

$|x+1| = 0$

$\Rightarrow x = -1$

☐Maximum value of $g = g(-1) = -|-1+1|+3 = 3$

Hence, function g does not have a minimum value.

(iii) $h(x) = \sin 2x + 5$

We know that $-1 \le \sin 2x \le 1$.

□ $-1 + 5 \le \sin 2x + 5 \le 1 + 5$

□ $4 \le \sin 2x + 5 \le 6$

Hence, the maximum and minimum values of h are 6 and 4 respectively.

(iv) $f(x) = |\sin 4x + 3|$

We know that $-1 \le \sin 4x \le 1$.

□ $2 \le \sin 4x + 3 \le 4$

□ $2 \le |\sin 4x + 3| \le 4$

Hence, the maximum and minimum values of f are 4 and 2 respectively.

(v) $h(x) = x + 1$, x □ $(-1, 1)$

Here, if a point x_0 is closest to -1, then we find $\frac{x_0}{2} + 1 < x_0 + 1$ for all x_0 □ $(-1, 1)$.

Also, if x_1 is closest to 1, then $x_1 + 1 < \frac{x_1 + 1}{2} + 1$ for all x_1 □ $(-1, 1)$.

Hence, function $h(x)$ has neither maximum nor minimum value in $(-1, 1)$.

Question 3:

Find the local maxima and local minima, if any, of the following functions. Find also the local maximum and the local minimum values, as the case may be:

(i). $f(x) = x^2$ (ii). $g(x) = x^3 - 3x$

(iii). $h(x) = \sin x + \cos$, $0 < x < \frac{\pi}{2}$ (iv). $f(x) = \sin x - \cos x$, $0 < x < 2\pi$

(v). $f(x) = x^3 - 6x^2 + 9x + 15$

(vi). $g(x) = \frac{x}{2} + \frac{2}{x}, x > 0$

(vii). $g(x) = \frac{1}{x^2 + 2}$

(viii). $f(x) = x\sqrt{1 - x}, x > 0$

Answer

(i) $f(x) = x^2$

$\therefore f'(x) = 2x$

Now,

$f'(x) = 0 \Rightarrow x = 0$

Thus, $x = 0$ is the only critical point which could possibly be the point of local maxima or local minima of f.

We have $f''(0) = 2$, which is positive.

Therefore, by second derivative test, $x = 0$ is a point of local minima and local minimum value of f at $x = 0$ is $f(0) = 0$.

(ii) $g(x) = x^3 - 3x$

$\therefore g'(x) = 3x^2 - 3$

Now,

$g'(x) = 0 \Rightarrow 3x^2 = 3 \Rightarrow x = \pm 1$

$g'(x) = 6x$

$g'(1) = 6 > 0$

$g'(-1) = -6 < 0$

By second derivative test, $x = 1$ is a point of local minima and local minimum value of g at $x = 1$ is $g(1) = 1^3 - 3 = 1 - 3 = -2$. However,

$x = -1$ is a point of local maxima and local maximum value of g at

$x = -1$ is $g(1) = (-1)^3 - 3(-1) = -1 + 3 = 2$.

(iii) $h(x) = \sin x + \cos x$, $0 < x < \frac{\pi}{2}$

$\therefore h'(x) = \cos x - \sin x$

$h'(x) = 0 \Rightarrow \sin x = \cos x \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$

$h''(x) = -\sin x - \cos x = -(\sin x + \cos x)$

$h''\left(\frac{\pi}{4}\right) = -\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = -\frac{2}{\sqrt{2}} = -\sqrt{2} < 0$

Therefore, by second derivative test, $x = \frac{\pi}{4}$ is a point of local maxima and the local maximum value of h at $x = \frac{\pi}{4}$ is $h\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}.$

(iv) $f(x) = \sin x - \cos x$, $0 < x < 2\pi$

$\therefore f'(x) = \cos x + \sin x$

$f'(x) = 0 \Rightarrow \cos x = -\sin x \Rightarrow \tan x = -1 \Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4} \in (0, 2\pi)$

$f''(x) = -\sin x + \cos x$

$f''\left(\frac{3\pi}{4}\right) = -\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2} > 0$

$f''\left(\frac{7\pi}{4}\right) = -\sin\frac{7\pi}{4} + \cos\frac{7\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2} > 0$

Therefore, by second derivative test, $x = \frac{3\pi}{4}$ is a point of local maxima and the local maximum value of f at $x = \frac{3\pi}{4}$ is

$f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} - \cos\frac{3\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}.$ However, $x = \frac{7\pi}{4}$ is a point of local minima and the local minimum value of f at $x = \frac{7\pi}{4}$ is $f\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{4} - \cos\frac{7\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$.

(v) $f(x) = x^3 - 6x^2 + 9x + 15$

$\therefore f'(x) = 3x^2 - 12x + 9$

$f'(x) = 0 \Rightarrow 3(x^2 - 4x + 3) = 0$

$\Rightarrow 3(x-1)(x-3) = 0$

$\Rightarrow x = 1, 3$

Now, f''

$(x) = 6x - 12 = 6(x-2)$

$f''(1) = 6(1-2) = -6 < 0$

$f''(3) = 6(3-2) = 6 > 0$

Therefore, by second derivative test, $x = 1$ is a point of local maxima and the local maximum value of f at $x = 1$ is $f(1) = 1 - 6 + 9 + 15 = 19$. However, $x = 3$ is a point of local minima and the local minimum value of f at $x = 3$ is $f(3) = 27 - 54 + 27 + 15 = 15$.

(vi) $g(x)=\frac{x}{2}+\frac{2}{x}, x>0$

$$\therefore g'(x)=\frac{1}{2}-\frac{2}{x^2}$$

Now,

$$g'(x)=0 \text{ gives } \frac{2}{x^2}=\frac{1}{2} \Rightarrow x^2=4 \Rightarrow x=\pm 2$$

Since $x > 0$, we take $x = 2$.

Now,

$$g''(x)=\frac{4}{x^3}$$

$$g''(2)=\frac{4}{2^3}=\frac{1}{2}>0$$

Therefore, by second derivative test, $x = 2$ is a point of local minima and the local minimum value of g at $x = 2$ is $g(2) = \frac{2}{2}+\frac{2}{2}=1+1=2.$

(vii) $g(x)=\frac{1}{x^2+2}$

$$\therefore g'(x)=\frac{-(2x)}{(x^2+2)^2}$$

$$g'(x)=0 \Rightarrow \frac{-2x}{(x^2+2)^2}=0 \Rightarrow x=0$$

Now, for values close to $x = 0$ and to the left of 0, $g'(x)>0.$ Also, for values close to $x = 0$ and to the right of 0, $g'(x)<0$.

Therefore, by first derivative test, $x = 0$ is a point of local maxima and the local maximum value of $g(0)$ is $\frac{1}{0+2} = \frac{1}{2}$.

(viii) $f(x) = x\sqrt{1-x},\ x > 0$

$$\therefore f'(x) = \sqrt{1-x} + x \cdot \frac{1}{2\sqrt{1-x}}(-1) = \sqrt{1-x} - \frac{x}{2\sqrt{1-x}}$$

$$= \frac{2(1-x)-x}{2\sqrt{1-x}} = \frac{2-3x}{2\sqrt{1-x}}$$

$$f'(x) = 0 \Rightarrow \frac{2-3x}{2\sqrt{1-x}} = 0 \Rightarrow 2-3x = 0 \Rightarrow x = \frac{2}{3}$$

$$f''(x) = \frac{1}{2}\left[\frac{\sqrt{1-x}(-3)-(2-3x)\left(\frac{-1}{2\sqrt{1-x}}\right)}{1-x}\right]$$

$$= \frac{\sqrt{1-x}(-3)+(2-3x)\left(\frac{1}{2\sqrt{1-x}}\right)}{2(1-x)}$$

$$= \frac{-6(1-x)+(2-3x)}{4(1-x)^{\frac{3}{2}}}$$

$$= \frac{3x-4}{4(1-x)^{\frac{3}{2}}}$$

$$f''\left(\frac{2}{3}\right) = \frac{3\left(\frac{2}{3}\right)-4}{4\left(1-\frac{2}{3}\right)^{\frac{3}{2}}} = \frac{2-4}{4\left(\frac{1}{3}\right)^{\frac{3}{2}}} = \frac{-1}{2\left(\frac{1}{3}\right)^{\frac{3}{2}}} < 0$$

Therefore, by second derivative test, $x = \frac{2}{3}$ is a point of local maxima and the local maximum value of f at $x = \frac{2}{3}$ is

$$f\left(\frac{2}{3}\right)=\frac{2}{3}\sqrt{1-\frac{2}{3}}=\frac{2}{3}\sqrt{\frac{1}{3}}=\frac{2}{3\sqrt{3}}=\frac{2\sqrt{3}}{9}.$$

Question 4:

Prove that the following functions do not have maxima or minima:

(i) $f(x) = e^x$ (ii) $g(x) = \log x$

(iii) $h(x) = x^3 + x^2 + x + 1$

Answer

i. We have,

$f(x) = e^x$

$\therefore f'(x)=e^x$

Now, if $f'(x)=0$, then $e^x=0$. But, the exponential function can never assume 0 for any value of x.

Therefore, there does not exist c☐ **R** such that $f'(c)=0.$

Hence, function f does not have maxima or minima.

ii. We have,

$g(x) = \log x$

$\therefore g'(x)=\frac{1}{x}$

Since $\log x$ is defined for a positive number x, $g'(x)>0$ for any x.

Therefore, there does not exist c☐ **R** such that $g'(c)=0$.

Hence, function g does not have maxima or minima.

iii. We have,

$h(x) = x^3 + x^2 + x + 1$

$\therefore h'(x)=3x^2+2x+1$

Now,

$h(x) = 0$ ☐ $3x^2 + 2x + 1 = 0$ ☐ $x=\frac{-2\pm2\sqrt{2}i}{6}=\frac{-1\pm\sqrt{2}i}{3}\notin \mathbf{R}$

Therefore, there does not exist c☐ **R** such that $h'(c)=0$.

Hence, function *h* does not have maxima or minima.

Question 5:

Find the absolute maximum value and the absolute minimum value of the following functions in the given intervals:

(i) $f(x)=x^3, x\in[-2,2]$ (ii) $f(x)=\sin x+\cos x, x\in[0,\pi]$

(iii) $f(x)=4x-\frac{1}{2}x^2, x\in\left[-2,\frac{9}{2}\right]$

(iv) $f(x)=(x-1)^2+3, x\in[-3,1]$

Answer

(i) The given function is *f*(*x*) = x^3.

$\therefore f'(x)=3x^2$

Now,

$f'(x)=0 \Rightarrow x=0$

Then, we evaluate the value of *f* at critical point *x* = 0 and at end points of the interval [−2, 2].

f(0) = 0

f(−2) = $(-2)^3$ = −8

f(2) = $(2)^3$ = 8

Hence, we can conclude that the absolute maximum value of *f* on [−2, 2] is 8 occurring at *x* = 2. Also, the absolute minimum value of *f* on [−2, 2] is −8 occurring at *x* = −2.

(ii) The given function is *f*(*x*) = sin *x* + cos *x*.

$\therefore f'(x)=\cos x-\sin x$

Now,

$f'(x)=0 \Rightarrow \sin x=\cos x \Rightarrow \tan x=1 \Rightarrow x=\frac{\pi}{4}$

Then, we evaluate the value of *f* at critical point $x=\frac{\pi}{4}$ and at the end points of the interval [0, π].

$$f\left(\frac{\pi}{4}\right)=\sin\frac{\pi}{4}+\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\frac{2}{\sqrt{2}}=\sqrt{2}$$

$$f(0)=\sin 0+\cos 0=0+1=1$$

$$f(\pi)=\sin\pi+\cos\pi=0-1=-1$$

Hence, we can conclude that the absolute maximum value of f on [0, π] is $\sqrt{2}$ occurring at $x=\frac{\pi}{4}$ and the absolute minimum value of f on [0, π] is −1 occurring at x = π.

(iii) The given function is $f(x)=4x-\frac{1}{2}x^2.$

$$\therefore f'(x)=4-\frac{1}{2}(2x)=4-x$$

Now,

$$f'(x)=0 \Rightarrow x=4$$

Then, we evaluate the value of f at critical point x = 4 and at the end points of the interval $\left[-2, \frac{9}{2}\right]$.

$$f(4)=16-\frac{1}{2}(16)=16-8=8$$

$$f(-2)=-8-\frac{1}{2}(4)=-8-2=-10$$

$$f\left(\frac{9}{2}\right)=4\left(\frac{9}{2}\right)-\frac{1}{2}\left(\frac{9}{2}\right)^2=18-\frac{81}{8}=18-10.125=7.875$$

Hence, we can conclude that the absolute maximum value of f on $\left[-2, \frac{9}{2}\right]$ is 8 occurring at x = 4 and the absolute minimum value of f on $\left[-2, \frac{9}{2}\right]$ is −10 occurring at x = −2.

(iv) The given function is $f(x)=(x-1)^2+3.$

$$\therefore f'(x)=2(x-1)$$

Now,

$f'(x)=0 \Rightarrow 2(x - 1) = 0$ □ $x = 1$

Then, we evaluate the value of f at critical point $x = 1$ and at the end points of the interval [−3, 1].

$$f(1)=(1-1)^2+3=0+3=3$$

$$f(-3)=(-3-1)^2+3=16+3=19$$

Hence, we can conclude that the absolute maximum value of f on [−3, 1] is 19 occurring at $x = -3$ and the minimum value of f on [−3, 1] is 3 occurring at $x = 1$.

Question 6:

Find the maximum profit that a company can make, if the profit function is given by $p(x) = 41 - 24x - 18x^2$

Answer

The profit function is given as $p(x) = 41 - 24x - 18x^2$.

$$\therefore p'(x)=-24-36x$$

$$p''(x)=-36$$

Now,

$$p'(x)=0 \Rightarrow x=\frac{-24}{36}=-\frac{2}{3}$$

Also,

$$p''\left(\frac{-2}{3}\right)=-36<0$$

By second derivative test, $x=-\frac{2}{3}$ is the point of local maxima of p.

$$\therefore \text{Maximum profit} = p\left(-\frac{2}{3}\right)$$

$$=41-24\left(-\frac{2}{3}\right)-18\left(-\frac{2}{3}\right)^2$$

$$=41+16-8$$

$$=49$$

Hence, the maximum profit that the company can make is 49 units.

Question 7:

Find the intervals in which the function *f* given by $f(x)=x^3+\frac{1}{x^3}, x\neq 0$ is

(i) increasing (ii) decreasing

Answer

$$f(x)=x^3+\frac{1}{x^3}$$

$$\therefore f'(x)=3x^2-\frac{3}{x^4}=\frac{3x^6-3}{x^4}$$

$$\text{Then, } f'(x)=0 \Rightarrow 3x^6-3=0 \Rightarrow x^6=1 \Rightarrow x=\pm 1$$

Now, the points $x = 1$ and $x = -1$ divide the real line into three disjoint intervals

i.e., $(-\infty,-1),(-1,1), \text{ and } (1,\infty).$

In intervals $(-\infty,-1) \text{ and } (1,\infty)$ i.e., when $x < -1$ and $x > 1$, $f'(x)>0.$

Thus, when $x < -1$ and $x > 1$, *f* is increasing.

In interval $(-1, 1)$ i.e., when $-1 < x < 1$, $f'(x)<0.$

Thus, when $-1 < x < 1$, *f* is decreasing.

Question 8:

At what points in the interval $[0, 2\pi]$, does the function $\sin 2x$ attain its maximum value?

Answer

Let $f(x) = \sin 2x$.

$$\therefore f'(x)=2\cos 2x$$

Now,

$$f'(x)=0 \Rightarrow \cos 2x=0$$

$$\Rightarrow 2x=\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$$

$$\Rightarrow x=\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

Then, we evaluate the values of *f* at critical points $x=\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ and at the end points of the interval [0, 2π].

$$f\left(\frac{\pi}{4}\right)=\sin\frac{\pi}{2}=1, f\left(\frac{3\pi}{4}\right)=\sin\frac{3\pi}{2}=-1$$

$$f\left(\frac{5\pi}{4}\right)=\sin\frac{5\pi}{2}=1, f\left(\frac{7\pi}{4}\right)=\sin\frac{7\pi}{2}=-1$$

$$f(0)=\sin 0=0, f(2\pi)=\sin 2\pi=0$$

Hence, we can conclude that the absolute maximum value of *f* on [0, 2π] is occurring at $x=\frac{\pi}{4}$ and $x=\frac{5\pi}{4}$.

Question 9:

What is the maximum value of the function sin *x* + cos *x*?

Answer

Let *f*(*x*) = sin *x* + cos *x*.

$$\therefore f'(x)=\cos x-\sin x$$

$$f'(x)=0 \Rightarrow \sin x=\cos x \Rightarrow \tan x=1 \Rightarrow x=\frac{\pi}{4}, \frac{5\pi}{4}...,$$

$$f''(x)=-\sin x-\cos x=-(\sin x+\cos x)$$

Now, $f''(x)$ will be negative when (sin *x* + cos *x*) is positive i.e., when sin *x* and cos *x* are both positive. Also, we know that sin *x* and cos *x* both are positive in the first quadrant. Then, $f''(x)$ will be negative when $x\in\left(0,\frac{\pi}{2}\right)$.

Thus, we consider $x=\frac{\pi}{4}$.

$$f''\left(\frac{\pi}{4}\right)=-\left(\sin\frac{\pi}{4}+\cos\frac{\pi}{4}\right)=-\left(\frac{2}{\sqrt{2}}\right)=-\sqrt{2}<0$$

□By second derivative test, f will be the maximum at $x = \frac{\pi}{4}$ and the maximum value of f is $f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$.

Question 10:

Find the maximum value of $2x^3 - 24x + 107$ in the interval [1, 3]. Find the maximum value of the same function in [−3, −1].

Answer

Let $f(x) = 2x^3 - 24x + 107$.

$\therefore f'(x) = 6x^2 - 24 = 6(x^2 - 4)$

Now,

$f'(x) = 0 \Rightarrow 6(x^2 - 4) = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

We first consider the interval [1, 3].

Then, we evaluate the value of f at the critical point x = 2 □ [1, 3] and at the end points of the interval [1, 3].

$f(2) = 2(8) - 24(2) + 107 = 16 - 48 + 107 = 75$

$f(1) = 2(1) - 24(1) + 107 = 2 - 24 + 107 = 85$

$f(3) = 2(27) - 24(3) + 107 = 54 - 72 + 107 = 89$

Hence, the absolute maximum value of $f(x)$ in the interval [1, 3] is 89 occurring at x = 3.

Next, we consider the interval [−3, −1].

Evaluate the value of f at the critical point x = −2 □ [−3, −1] and at the end points of the interval [1, 3].

$f(-3) = 2(-27) - 24(-3) + 107 = -54 + 72 + 107 = 125$

$f(-1) = 2(-1) - 24(-1) + 107 = -2 + 24 + 107 = 129$

$f(-2) = 2(-8) - 24(-2) + 107 = -16 + 48 + 107 = 139$

Hence, the absolute maximum value of $f(x)$ in the interval [−3, −1] is 139 occurring at x = −2.

Question 11:

It is given that at x = 1, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value, on the interval [0, 2]. Find the value of a.

Answer

Let $f(x) = x^4 - 62x^2 + ax + 9$.

$$\therefore f'(x) = 4x^3 - 124x + a$$

It is given that function f attains its maximum value on the interval [0, 2] at x = 1.

$$\therefore f'(1) = 0$$

$$\Rightarrow 4 - 124 + a = 0$$

$$\Rightarrow a = 120$$

Hence, the value of a is 120.

Question 12:

Find the maximum and minimum values of $x + \sin 2x$ on [0, 2π].

Answer

Let $f(x) = x + \sin 2x$.

$$\therefore f'(x) = 1 + 2\cos 2x$$

$$\text{Now, } f'(x) = 0 \Rightarrow \cos 2x = -\frac{1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$$

$$2x = 2n\pi \pm \frac{2\pi}{3}, \quad n \in \mathbf{Z}$$

$$\Rightarrow x = n\pi \pm \frac{\pi}{3}, \; n \in \mathbf{Z}$$

$$\Rightarrow x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3} \in [0, 2\pi]$$

Then, we evaluate the value of f at critical points $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ and at the end points of the interval [0, 2π].

$$f\left(\frac{\pi}{3}\right)=\frac{\pi}{3}+\sin\frac{2\pi}{3}=\frac{\pi}{3}+\frac{\sqrt{3}}{2}$$

$$f\left(\frac{2\pi}{3}\right)=\frac{2\pi}{3}+\sin\frac{4\pi}{3}=\frac{2\pi}{3}-\frac{\sqrt{3}}{2}$$

$$f\left(\frac{4\pi}{3}\right)=\frac{4\pi}{3}+\sin\frac{8\pi}{3}=\frac{4\pi}{3}+\frac{\sqrt{3}}{2}$$

$$f\left(\frac{5\pi}{3}\right)=\frac{5\pi}{3}+\sin\frac{10\pi}{3}=\frac{5\pi}{3}-\frac{\sqrt{3}}{2}$$

$$f(0)=0+\sin 0=0$$

$$f(2\pi)=2\pi+\sin 4\pi=2\pi+0=2\pi$$

Hence, we can conclude that the absolute maximum value of $f(x)$ in the interval $[0, 2\pi]$ is 2π occurring at $x = 2\pi$ and the absolute minimum value of $f(x)$ in the interval $[0, 2\pi]$ is 0 occurring at $x = 0$.

Question 13:

Find two numbers whose sum is 24 and whose product is as large as possible.

Answer

Let one number be x. Then, the other number is $(24 - x)$.

Let $P(x)$ denote the product of the two numbers. Thus, we have:

$$P(x)=x(24-x)=24x-x^2$$

$$\therefore P'(x)=24-2x$$

$$P''(x)=-2$$

Now,

$$P'(x)=0 \Rightarrow x=12$$

Also,

$$P''(12)=-2<0$$

☐By second derivative test, $x = 12$ is the point of local maxima of P. Hence, the product of the numbers is the maximum when the numbers are 12 and $24 - 12 = 12$.

Question 14:

Find two positive numbers x and y such that $x + y = 60$ and xy^3 is maximum.

Answer

The two numbers are x and y such that $x + y = 60$.

□ $y = 60 - x$

Let $f(x) = xy^3$.

$$\Rightarrow f(x) = x(60-x)^3$$

$$\therefore f'(x) = (60-x)^3 - 3x(60-x)^2$$
$$= (60-x)^2[60-x-3x]$$
$$= (60-x)^2(60-4x)$$

$$\text{And, } f''(x) = -2(60-x)(60-4x) - 4(60-x)^2$$
$$= -2(60-x)[60-4x+2(60-x)]$$
$$= -2(60-x)(180-6x)$$
$$= -12(60-x)(30-x)$$

$$\text{Now, } f'(x) = 0 \Rightarrow x = 60 \text{ or } x = 15$$

$$\text{When } x = 60, f''(x) = 0.$$

$$\text{When } x = 15, f''(x) = -12(60-15)(30-15) = -12 \times 45 \times 15 < 0.$$

□By second derivative test, $x = 15$ is a point of local maxima of f. Thus, function xy^3 is maximum when $x = 15$ and $y = 60 - 15 = 45$.

Hence, the required numbers are 15 and 45.

Question 15:

Find two positive numbers x and y such that their sum is 35 and the product x^2y^5 is a maximum

Answer

Let one number be x. Then, the other number is $y = (35 - x)$.

Let $P(x) = x^2y^5$. Then, we have:

$$P(x)=x^2(35-x)^5$$

$$\therefore P'(x)=2x(35-x)^5-5x^2(35-x)^4$$

$$=x(35-x)^4\left[2(35-x)-5x\right]$$

$$=x(35-x)^4(70-7x)$$

$$=7x(35-x)^4(10-x)$$

$$\text{And, } P''(x)=7(35-x)^4(10-x)+7x\left[-(35-x)^4-4(35-x)^3(10-x)\right]$$

$$=7(35-x)^4(10-x)-7x(35-x)^4-28x(35-x)^3(10-x)$$

$$=7(35-x)^3\left[(35-x)(10-x)-x(35-x)-4x(10-x)\right]$$

$$=7(35-x)^3\left[350-45x+x^2-35x+x^2-40x+4x^2\right]$$

$$=7(35-x)^3(6x^2-120x+350)$$

Now, $P'(x)=0 \Rightarrow$ $x = 0$, $x = 35$, $x = 10$

When $x = 35$, $f'(x)=f(x)=0$ and $y = 35 - 35 = 0$. This will make the product $x^2 y^5$ equal to 0.

When $x = 0$, $y = 35 - 0 = 35$ and the product x^2y^2 will be 0.

☐ $x = 0$ and $x = 35$ cannot be the possible values of x.

When $x = 10$, we have:

$$P''(x)=7(35-10)^3(6\times100-120\times10+350)$$

$$=7(25)^3(-250)<0$$

☐ By second derivative test, $P(x)$ will be the maximum when $x = 10$ and $y = 35 - 10 = 25$.

Hence, the required numbers are 10 and 25.

Question 16:

Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum.

Answer

Let one number be x. Then, the other number is $(16 - x)$.

Let the sum of the cubes of these numbers be denoted by $S(x)$. Then,

$$S(x) = x^3 + (16 - x)^3$$

$$\therefore S'(x) = 3x^2 - 3(16 - x)^2,\ S''(x) = 6x + 6(16 - x)$$

$$\text{Now, } S'(x) = 0 \Rightarrow 3x^2 - 3(16 - x)^2 = 0$$

$$\Rightarrow x^2 - (16 - x)^2 = 0$$

$$\Rightarrow x^2 - 256 - x^2 + 32x = 0$$

$$\Rightarrow x = \frac{256}{32} = 8$$

Now, $S''(8) = 6(8) + 6(16 - 8) = 48 + 48 = 96 > 0$

☐ By second derivative test, $x = 8$ is the point of local minima of S.

Hence, the sum of the cubes of the numbers is the minimum when the numbers are 8 and $16 - 8 = 8$.

Question 17:

A square piece of tin of side 18 cm is to made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum possible?

Answer

Let the side of the square to be cut off be x cm. Then, the length and the breadth of the box will be $(18 - 2x)$ cm each and the height of the box is x cm.

Therefore, the volume $V(x)$ of the box is given by,

$V(x) = x(18 - 2x)^2$

$$\therefore V'(x) = (18-2x)^2 - 4x(18-2x)$$
$$= (18-2x)[18-2x-4x]$$
$$= (18-2x)(18-6x)$$
$$= 6\times 2(9-x)(3-x)$$
$$= 12(9-x)(3-x)$$

And, $V''(x) = 12[-(9-x)-(3-x)]$
$$= -12(9-x+3-x)$$
$$= -12(12-2x)$$
$$= -24(6-x)$$

Now, $V'(x) = 0 \Rightarrow x = 9$ or $x = 3$

If $x = 9$, then the length and the breadth will become 0.

$\therefore x \neq 9$.

$\Rightarrow x = 3$.

Now, $V''(3) = -24(6-3) = -72 < 0$

$\therefore$ By second derivative test, $x = 3$ is the point of maxima of V.

Hence, if we remove a square of side 3 cm from each corner of the square tin and make a box from the remaining sheet, then the volume of the box obtained is the largest possible.

Question 18:

A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is the maximum possible?

Answer

Let the side of the square to be cut off be x cm. Then, the height of the box is x, the length is $45 - 2x$, and the breadth is $24 - 2x$.

Therefore, the volume $V(x)$ of the box is given by,

$$V(x) = x(45-2x)(24-2x)$$
$$= x(1080-90x-48x+4x^2)$$
$$= 4x^3 - 138x^2 + 1080x$$
$$\therefore V'(x) = 12x^2 - 276x + 1080$$
$$= 12(x^2 - 23x + 90)$$
$$= 12(x-18)(x-5)$$
$$V''(x) = 24x - 276 = 12(2x-23)$$

Now, $V'(x) = 0 \Rightarrow x = 18$ and $x = 5$

It is not possible to cut off a square of side 18 cm from each corner of the rectangular sheet. Thus, x cannot be equal to 18.

□$x = 5$

Now, $V''(5) = 12(10-23) = 12(-13) = -156 < 0$

$\therefore$ By second derivative test, $x = 5$ is the point of maxima.

Hence, the side of the square to be cut off to make the volume of the box maximum possible is 5 cm.

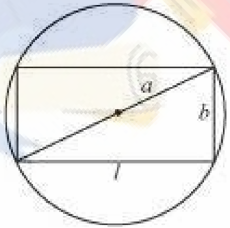
Question 19:

Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.

Answer

Let a rectangle of length *l* and breadth *b* be inscribed in the given circle of radius *a*.

Then, the diagonal passes through the centre and is of length 2*a* cm.



Now, by applying the Pythagoras theorem, we have:

$$(2a)^2 = l^2 + b^2$$

$$\Rightarrow b^2 = 4a^2 - l^2$$

$$\Rightarrow b = \sqrt{4a^2 - l^2}$$

□Area of the rectangle, $A = l\sqrt{4a^2 - l^2}$

$$\therefore \frac{dA}{dl} = \sqrt{4a^2 - l^2} + l\frac{1}{2\sqrt{4a^2 - l^2}}(-2l) = \sqrt{4a^2 - l^2} - \frac{l^2}{\sqrt{4a^2 - l^2}}$$

$$= \frac{4a^2 - 2l^2}{\sqrt{4a^2 - l^2}}$$

$$\frac{d^2A}{dl^2} = \frac{\sqrt{4a^2 - l^2}(-4l) - (4a^2 - 2l^2)\frac{(-2l)}{2\sqrt{4a^2 - l^2}}}{(4a^2 - l^2)}$$

$$= \frac{(4a^2 - l^2)(-4l) + l(4a^2 - 2l^2)}{(4a^2 - l^2)^{\frac{3}{2}}}$$

$$= \frac{-12a^2l + 2l^3}{(4a^2 - l^2)^{\frac{3}{2}}} = \frac{-2l(6a^2 - l^2)}{(4a^2 - l^2)^{\frac{3}{2}}}$$

$$\text{Now,} \frac{dA}{dl} = 0 \text{ gives } 4a^2 = 2l^2 \Rightarrow l = \sqrt{2}a$$

$$\Rightarrow b = \sqrt{4a^2 - 2a^2} = \sqrt{2a^2} = \sqrt{2}a$$

Now, when $l = \sqrt{2}a$,

$$\frac{d^2\text{A}}{dl^2} = \frac{-2(\sqrt{2}a)(6a^2 - 2a^2)}{2\sqrt{2}a^3} = \frac{-8\sqrt{2}a^3}{2\sqrt{2}a^3} = -4 < 0$$

∴ By the second derivative test, when $l = \sqrt{2}a$, then the area of the rectangle is the maximum.

Since $l = b = \sqrt{2}a$, the rectangle is a square.

Hence, it has been proved that of all the rectangles inscribed in the given fixed circle, the square has the maximum area.

Question 20:

Show that the right circular cylinder of given surface and maximum volume is such that is heights is equal to the diameter of the base.

Answer

Let *r* and *h* be the radius and height of the cylinder respectively.

Then, the surface area (*S*) of the cylinder is given by,

$$S = 2\pi r^2 + 2\pi rh$$

$$\Rightarrow h = \frac{S - 2\pi r^2}{2\pi r}$$

$$= \frac{S}{2\pi}\left(\frac{1}{r}\right) - r$$

Let *V* be the volume of the cylinder. Then,

$$V = \pi r^2 h = \pi r^2\left[\frac{S}{2\pi}\left(\frac{1}{r}\right) - r\right] = \frac{Sr}{2} - \pi r^3$$

$$\text{Then, } \frac{dV}{dr} = \frac{S}{2} - 3\pi r^2, \ \frac{d^2V}{dr^2} = -6\pi r$$

$$\text{Now, } \frac{dV}{dr} = 0 \Rightarrow \frac{S}{2} = 3\pi r^2 \Rightarrow r^2 = \frac{S}{6\pi}$$

$$\text{When } r^2 = \frac{S}{6\pi}, \text{ then } \frac{d^2V}{dr^2} = -6\pi\left(\sqrt{\frac{S}{6\pi}}\right) < 0.$$

☐ By second derivative test, the volume is the maximum when $r^2 = \frac{S}{6\pi}$.

$$\text{Now, when } r^2 = \frac{S}{6\pi}, \text{ then } h = \frac{6\pi r^2}{2\pi}\left(\frac{1}{r}\right) - r = 3r - r = 2r.$$

Hence, the volume is the maximum when the height is twice the radius i.e., when the height is equal to the diameter.

Question 21:

Of all the closed cylindrical cans (right circular), of a given volume of 100 cubic centimetres, find the dimensions of the can which has the minimum surface area?

Answer

Let *r* and *h* be the radius and height of the cylinder respectively.

Then, volume (*V*) of the cylinder is given by,

$$V = \pi r^2 h = 100 \qquad \text{(given)}$$

$$\therefore h = \frac{100}{\pi r^2}$$

Surface area (*S*) of the cylinder is given by,

$$S = 2\pi r^2 + 2\pi r h = 2\pi r^2 + \frac{200}{r}$$

$$\therefore \frac{dS}{dr} = 4\pi r - \frac{200}{r^2}, \quad \frac{d^2S}{dr^2} = 4\pi + \frac{400}{r^3}$$

$$\frac{dS}{dr} = 0 \Rightarrow 4\pi r = \frac{200}{r^2}$$

$$\Rightarrow r^3 = \frac{200}{4\pi} = \frac{50}{\pi}$$

$$\Rightarrow r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$$

Now, it is observed that when $r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$, $\frac{d^2S}{dr^2} > 0$.

□By second derivative test, the surface area is the minimum when the radius of the cylinder is $\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm.

$$\text{When } r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}, \ h = \frac{100}{\pi\left(\frac{50}{\pi}\right)^{\frac{2}{3}}} = \frac{2\times 50}{(50)^{\frac{2}{3}}(\pi)^{1-\frac{2}{3}}} = 2\left(\frac{50}{\pi}\right)^{\frac{1}{3}} \text{ cm.}$$

Hence, the required dimensions of the can which has the minimum surface area is given by radius = $\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm and height = $2\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm.

Question 22:

A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum?

Answer

Let a piece of length l be cut from the given wire to make a square.

Then, the other piece of wire to be made into a circle is of length $(28 - l)$ m.

Now, side of square = $\frac{l}{4}$.

Let r be the radius of the circle. Then, $2\pi r = 28 - l \Rightarrow r = \frac{1}{2\pi}(28 - l)$.

The combined areas of the square and the circle (A) is given by,

$$A = (\text{side of the square})^2 + \pi r^2$$

$$= \frac{l^2}{16} + \pi\left[\frac{1}{2\pi}(28 - l)\right]^2$$

$$= \frac{l^2}{16} + \frac{1}{4\pi}(28 - l)^2$$

$$\therefore \frac{dA}{dl} = \frac{2l}{16} + \frac{2}{4\pi}(28 - l)(-1) = \frac{l}{8} - \frac{1}{2\pi}(28 - l)$$

$$\frac{d^2A}{dl^2} = \frac{1}{8} + \frac{1}{2\pi} > 0$$

$$\text{Now, } \frac{dA}{dl} = 0 \Rightarrow \frac{l}{8} - \frac{1}{2\pi}(28 - l) = 0$$

$$\Rightarrow \frac{\pi l - 4(28 - l)}{8\pi} = 0$$

$$\Rightarrow (\pi + 4)l - 112 = 0$$

$$\Rightarrow l = \frac{112}{\pi + 4}$$

Thus, when $l = \frac{112}{\pi + 4}, \frac{d^2A}{dl^2} > 0.$

$\therefore$ By second derivative test, the area (A) is the minimum when $l = \frac{112}{\pi + 4}$.

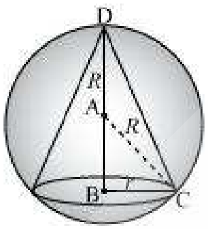
Hence, the combined area is the minimum when the length of the wire in making the square is $\frac{112}{\pi+4}$ cm while the length of the wire in making the circle is $28-\frac{112}{\pi+4}=\frac{28\pi}{\pi+4}$ cm.

Question 23:

Prove that the volume of the largest cone that can be inscribed in a sphere of radius *R* is $\frac{8}{27}$ of the volume of the sphere.

Answer

Let *r* and *h* be the radius and height of the cone respectively inscribed in a sphere of radius *R*.



Let *V* be the volume of the cone.

Then, $V=\frac{1}{3}\pi r^2 h$

Height of the cone is given by,

$h = R + AB = R+\sqrt{R^2-r^2}$ [ABC is a right triangle]

$$\therefore V = \frac{1}{3}\pi r^2\left(R + \sqrt{R^2 - r^2}\right)$$

$$= \frac{1}{3}\pi r^2 R + \frac{1}{3}\pi r^2\sqrt{R^2 - r^2}$$

$$\therefore \frac{dV}{dr} = \frac{2}{3}\pi r R + \frac{2}{3}\pi r\sqrt{R^2 - r^2} + \frac{1}{3}\pi r^2 \cdot \frac{(-2r)}{2\sqrt{R^2 - r^2}}$$

$$= \frac{2}{3}\pi r R + \frac{2}{3}\pi r\sqrt{R^2 - r^2} - \frac{1}{3}\pi\frac{r^3}{\sqrt{R^2 - r^2}}$$

$$= \frac{2}{3}\pi r R + \frac{2\pi r\left(R^2 - r^2\right) - \pi r^3}{3\sqrt{R^2 - r^2}}$$

$$= \frac{2}{3}\pi r R + \frac{2\pi r R^2 - 3\pi r^3}{3\sqrt{R^2 - r^2}}$$

$$\frac{d^2V}{dr^2} = \frac{2\pi R}{3} + \frac{3\sqrt{R^2 - r^2}\left(2\pi R^2 - 9\pi r^2\right) - \left(2\pi r R^2 - 3\pi r^3\right)\cdot\frac{(-2r)}{6\sqrt{R^2 - r^2}}}{9\left(R^2 - r^2\right)}$$

$$= \frac{2}{3}\pi R + \frac{9\left(R^2 - r^2\right)\left(2\pi R^2 - 9\pi r^2\right) + 2\pi r^2 R^2 + 3\pi r^4}{27\left(R^2 - r^2\right)^{\frac{3}{2}}}$$

$$\text{Now, } \frac{dV}{dr} = 0 \Rightarrow \frac{2}{3}\pi rR = \frac{3\pi r^3 - 2\pi r R^2}{3\sqrt{R^2 - r^2}}$$

$$\Rightarrow 2R = \frac{3r^2 - 2R^2}{\sqrt{R^2 - r^2}} \Rightarrow 2R\sqrt{R^2 - r^2} = 3r^2 - 2R^2$$

$$\Rightarrow 4R^2\left(R^2 - r^2\right) = \left(3r^2 - 2R^2\right)^2$$

$$\Rightarrow 4R^4 - 4R^2 r^2 = 9r^4 + 4R^4 - 12r^2 R^2$$

$$\Rightarrow 9r^4 = 8R^2 r^2$$

$$\Rightarrow r^2 = \frac{8}{9}R^2$$

When $r^2 = \frac{8}{9}R^2$, then $\frac{d^2V}{dr^2} < 0$.

☐ By second derivative test, the volume of the cone is the maximum when $r^2 = \frac{8}{9}R^2$.

When $r^2=\frac{8}{9}R^2,\ h=R+\sqrt{R^2-\frac{8}{9}R^2}=R+\sqrt{\frac{1}{9}R^2}=R+\frac{R}{3}=\frac{4}{3}R.$

Therefore,

$$=\frac{1}{3}\pi\left(\frac{8}{9}R^2\right)\left(\frac{4}{3}R\right)$$

$$=\frac{8}{27}\left(\frac{4}{3}\pi R^3\right)$$

$$=\frac{8}{27}\times(\text{Volume of the sphere})$$

Hence, the volume of the largest cone that can be inscribed in the sphere is $\frac{8}{27}$ the volume of the sphere.

Question 24:

Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ time the radius of the base.

Answer

Let *r* and *h* be the radius and the height (altitude) of the cone respectively.

Then, the volume (*V)* of the cone is given as:

$$V=\frac{1}{3\pi}\pi r^2 h\Rightarrow h=\frac{3V}{r^2}$$

The surface area (*S)* of the cone is given by,

S = π*rl* (where *l* is the slant height)

$$=\pi r\sqrt{r^2+h^2}$$

$$=\pi r\sqrt{r^2+\frac{9V^2}{\pi^2 r^4}}=\frac{\pi r\sqrt{9\pi^2 r^6+V^2}}{\pi r^2}$$

$$=\frac{1}{r}\sqrt{\pi^2 r^6+9V^2}$$

$$\therefore \frac{dS}{dr} = \frac{r \cdot \dfrac{6\pi^2 r^5}{2\sqrt{\pi^2 r^6 + 9V^2}} - \sqrt{\pi^2 r^6 + 9V^2}}{r^2}$$

$$= \frac{3\pi^2 r^6 - \pi^2 r^6 - 9V^2}{r^2\sqrt{\pi^2 r^6 + 9V^2}}$$

$$= \frac{2\pi^2 r^6 - 9V^2}{r^2\sqrt{\pi^2 r^6 + 9V^2}}$$

$$= \frac{2\pi^2 r^6 - 9V^2}{r^2\sqrt{\pi^2 r^6 + 9V^2}}$$

$$\text{Now}, \frac{dS}{dr} = 0 \Rightarrow 2\pi^2 r^6 = 9V^2 \Rightarrow r^6 = \frac{9V^2}{2\pi^2}$$

Thus, it can be easily verified that when $r^6 = \frac{9V^2}{2\pi^2}, \frac{d^2S}{dr^2} > 0.$

☐ By second derivative test, the surface area of the cone is the least when $r^6 = \frac{9V^2}{2\pi^2}.$

$$\text{When } r^6 = \frac{9V^2}{2\pi^2},\ h = \frac{3V}{\pi r^2} = \frac{3}{\pi r^2}\left(\frac{2\pi^2 r^6}{9}\right)^{\frac{1}{2}} = \frac{3}{\pi r^2} \cdot \frac{\sqrt{2}\pi r^3}{3} = \sqrt{2}r.$$

Hence, for a given volume, the right circular cone of the least curved surface has an altitude equal to $\sqrt{2}$ times the radius of the base.

Question 25:

Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.

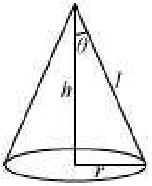
Answer

Let θ be the semi-vertical angle of the cone.

It is clear that $\theta \in \left[0, \frac{\pi}{2}\right].$

Let r, h, and l be the radius, height, and the slant height of the cone respectively.

The slant height of the cone is given as constant.



Now, $r = l \sin \theta$ and $h = l \cos \theta$

The volume (V) of the cone is given by,

$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi\left(l^2 \sin^2 \theta\right)\left(l \cos\theta\right)$$
$$= \frac{1}{3}\pi l^3 \sin^2 \theta \cos\theta$$
$$\therefore \frac{dV}{d\theta} = \frac{l^3\pi}{3}\left[\sin^2\theta(-\sin\theta) + \cos\theta(2\sin\theta\cos\theta)\right]$$
$$= \frac{l^3\pi}{3}\left[-\sin^3 + 2\sin\theta\cos^2\theta\right]$$
$$\frac{d^2V}{d\theta^2} = \frac{l^3\pi}{3}\left[-3\sin^2\theta\cos\theta + 2\cos^3\theta - 4\sin^2\theta\cos\theta\right]$$
$$= \frac{l^3\pi}{3}\left[2\cos^3\theta - 7\sin^2\theta\cos\theta\right]$$

$$\text{Now,} \frac{dV}{d\theta} = 0$$
$$\Rightarrow \sin^3\theta = 2\sin\theta\cos^2\theta$$
$$\Rightarrow \tan^2\theta = 2$$
$$\Rightarrow \tan\theta = \sqrt{2}$$
$$\Rightarrow \theta = \tan^{-1}\sqrt{2}$$

Now, when $\theta = \tan^{-1}\sqrt{2}$, then $\tan^2\theta = 2$ or $\sin^2\theta = 2\cos^2\theta$.

Then, we have:

$$\frac{d^2V}{d\theta^2} = \frac{l^3\pi}{3}\left[2\cos^3\theta - 14\cos^3\theta\right] = -4\pi l^3\cos^3\theta < 0 \text{ for } \theta \in \left[0, \frac{\pi}{2}\right]$$

□By second derivative test, the volume (V) is the maximum when $\theta = \tan^{-1}\sqrt{2}$.

Hence, for a given slant height, the semi-vertical angle of the cone of the maximum volume is $\tan^{-1}\sqrt{2}$.

Question 27:

The point on the curve $x^2 = 2y$ which is nearest to the point (0, 5) is

(A) $(2\sqrt{2},4)$ (B) $(2\sqrt{2},0)$

(C) (0, 0) (D) (2, 2)

Answer

The given curve is $x^2 = 2y$.

For each value of x, the position of the point will be $\left(x, \frac{x^2}{2}\right)$.

The distance $d(x)$ between the points $\left(x, \frac{x^2}{2}\right)$ and (0, 5) is given by,

$$d(x)=\sqrt{(x-0)^2+\left(\frac{x^2}{2}-5\right)^2}=\sqrt{x^2+\frac{x^4}{4}+25-5x^2}=\sqrt{\frac{x^4}{4}-4x^2+25}$$

$$\therefore d'(x)=\frac{\left(x^3-8x\right)}{2\sqrt{\frac{x^4}{4}-4x^2+25}}=\frac{\left(x^3-8x\right)}{\sqrt{x^4-16x^2+100}}$$

$$\text{Now, } d'(x)=0 \Rightarrow x^3-8x=0$$

$$\Rightarrow x\left(x^2-8\right)=0$$

$$\Rightarrow x=0,\pm 2\sqrt{2}$$

$$\text{And}, d''(x) = \frac{\sqrt{x^4 - 16x^2 + 100}\left(3x^2 - 8\right) - \left(x^3 - 8x\right) \cdot \dfrac{4x^3 - 32x}{2\sqrt{x^4 - 16x^2 + 100}}}{\left(x^4 - 16x^2 + 100\right)}$$

$$= \frac{\left(x^4 - 16x^2 + 100\right)\left(3x^2 - 8\right) - 2\left(x^3 - 8x\right)\left(x^3 - 8x\right)}{\left(x^4 - 16x^2 + 100\right)^{\frac{3}{2}}}$$

$$= \frac{\left(x^4 - 16x^2 + 100\right)\left(3x^2 - 8\right) - 2\left(x^3 - 8x\right)^2}{\left(x^4 - 16x^2 + 100\right)^{\frac{3}{2}}}$$

When, $x = 0, \text{then } d''(x) = \dfrac{36(-8)}{6^3} < 0.$

When, $x = \pm 2\sqrt{2},\ d''(x) > 0.$

□By second derivative test, $d(x)$ is the minimum at $x = \pm 2\sqrt{2}$.

When $x = \pm 2\sqrt{2}, y = \dfrac{\left(2\sqrt{2}\right)^2}{2} = 4.$

Hence, the point on the curve $x^2 = 2y$ which is nearest to the point (0, 5) is $\left(\pm 2\sqrt{2},\ 4\right)$.

The correct answer is A.

Question 28:

For all real values of x, the minimum value of $\dfrac{1 - x + x^2}{1 + x + x^2}$ is

(A) 0 (B) 1

(C) 3 (D) $\dfrac{1}{3}$

Answer

Let $f(x) = \dfrac{1 - x + x^2}{1 + x + x^2}.$

$$\therefore f'(x)=\frac{(1+x+x^2)(-1+2x)-(1-x+x^2)(1+2x)}{(1+x+x^2)^2}$$

$$=\frac{-1+2x-x+2x^2-x^2+2x^3-1-2x+x+2x^2-x^2-2x^3}{(1+x+x^2)^2}$$

$$=\frac{2x^2-2}{(1+x+x^2)^2}=\frac{2(x^2-1)}{(1+x+x^2)^2}$$

$$\therefore f'(x)=0 \Rightarrow x^2=1\Rightarrow x=\pm 1$$

$$\text{Now, } f''(x)=\frac{2\left[(1+x+x^2)^2(2x)-(x^2-1)(2)(1+x+x^2)(1+2x)\right]}{(1+x+x^2)^4}$$

$$=\frac{4(1+x+x^2)\left[(1+x+x^2)x-(x^2-1)(1+2x)\right]}{(1+x+x^2)^4}$$

$$=\frac{4\left[x+x^2+x^3-x^2-2x^3+1+2x\right]}{(1+x+x^2)^3}$$

$$=\frac{4(1+3x-x^3)}{(1+x+x^2)^3}$$

$$\text{And, } f''(1)=\frac{4(1+3-1)}{(1+1+1)^3}=\frac{4(3)}{(3)^3}=\frac{4}{9}>0$$

$$\text{Also, } f''(-1)=\frac{4(1-3+1)}{(1-1+1)^3}=4(-1)=-4<0$$

□By second derivative test, *f* is the minimum at $x = 1$ and the minimum value is given by $f(1)=\frac{1-1+1}{1+1+1}=\frac{1}{3}$.

The correct answer is D.

Question 29:

The maximum value of $\left[x(x-1)+1\right]^{\frac{1}{3}}, 0\le x\le 1$ is

(A) $\left(\frac{1}{3}\right)^{\frac{1}{3}}$ (B) $\frac{1}{2}$

(C) 1 (D) 0

Answer

Let $f(x)=\left[x(x-1)+1\right]^{\frac{1}{3}}$.

$$\therefore f'(x)=\frac{2x-1}{3\left[x(x-1)+1\right]^{\frac{2}{3}}}$$

$$\text{Now, } f'(x)=0 \Rightarrow x=\frac{1}{2}$$

Then, we evaluate the value of *f* at critical point $x=\frac{1}{2}$ and at the end points of the interval [0, 1] {i.e., at *x* = 0 and *x* = 1}.

$$f(0)=\left[0(0-1)+1\right]^{\frac{1}{3}}=1$$

$$f(1)=\left[1(1-1)+1\right]^{\frac{1}{3}}=1$$

$$f\left(\frac{1}{2}\right)=\left[\frac{1}{2}\left(\frac{-1}{2}\right)+1\right]^{\frac{1}{3}}=\left(\frac{3}{4}\right)^{\frac{1}{3}}$$

Hence, we can conclude that the maximum value of *f* in the interval [0, 1] is 1.

The correct answer is C.