Exercise - 5.7

Find the second order derivatives of the functions given in Exercise 1 to 10.

Q1. $x^2 + 3x + 2$

A.1. Let y = x2 + 3x + 2

So, $\frac{dy}{dx} = 2x + 3 + 0$ (differentiation w r t 'x')

$\Rightarrow \frac{d^2y}{dx^2} = 2 + 0$ (Again " ") = 2

Q2. x^{20}

A.2. Let $y = x^{20}$

So, $\frac{dy}{dx} = 20x^{20-1} = 20x^{19}$

$\therefore \frac{d^2y}{dx^2} = 20 \times 19x^{19-1}$

$= 380x^{18}$

Q3. $x \,.\, \cos x$

A.3. Let $y = x\cos x$

So, $\frac{dy}{dx} = x\frac{d}{dx}\cos x + \frac{dx}{dx}\cos x$

$= -x\sin x + \cos x$

$$\therefore \frac{d^2y}{dx^2} = -x\frac{d}{dx}\sin x - \sin x\frac{dx}{dx} + \frac{d}{dx}\cos x$$

$$= -x\cos x - \sin x + (-\sin x)$$

$$= -(x\cos x + 2\sin x)$$

Q4. $\log\ x$

A.4. Let $y = \log x$

So, $\frac{dy}{dx} = \frac{d}{dx}\log x = \frac{1}{x}$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\frac{1}{x} = \frac{d}{dx}x^{-1} = -1x^{-1-1} = \frac{-1}{x^2}$$

Q5. $x^3 \log\ x$

A.5. Let $y = x^3 \log x$

So, $\frac{dy}{dx} = x^3\frac{d}{dx}\log x + \log 2.\frac{dx^3}{dx}$

$$= x^3.\frac{1}{x} + \log x \cdot 3x^2$$

$$= x^2 + \log x\left(3x^2\right)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(x^2 + \log x \cdot 3x^2\right)$$

$$= \frac{dx^2}{dx} + \log x\frac{d}{dx}\left(3x^2\right) + 3x^2\frac{d}{dx}\log x$$

$$= 2x + 6x\log x + 3x^2 \times \frac{1}{x}$$

$= 2x + 6x \log x + 3x$

$= 5x + 6x \log x$

Q6. $e^x \ \sin 5x$

A.6. Let $y = e^x \sin 5x$

so, $\frac{dy}{dx} = e^x \frac{d}{dx} \sin 5x + \sin 5x \frac{d}{dx} e^x$

$= e^x \cdot \cos 5x \frac{d}{dx}(5x) + e^x \sin 5x$

$= 5e^x \cos 5x + e^x \sin 5x.$

$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(5e^x \cos 5x + e^x \sin 5x\right)$

$= 5e^x \frac{d}{dx} \cos 5x + 5 \cos 5x \frac{d}{dx} e^x + e^x \frac{d}{dx} \sin 5x + \sin 5x \frac{d}{dx} e^x$

$= -5e^x \sin 5x \frac{d}{dx}(5x) + 5e^x \cos 5x + e^x \cos 5x \frac{d}{dx}(5x) + e^x \sin 5x$

$= -25e^x \sin 5x + 5e^x \cos 5x + 5e^x \cos 5x + e^x \sin 5x$

$= e^x (10 \cdot \cos 5x - 24 \sin 5x)$

$= 2e^x (5 \cos 5x - 12 \sin 5x)$

Q7. $e^{6x} \ \cos 3x$

A.7. Let $y = e^{6x} \cos 3x$

So, $\frac{dy}{dx} = e^{6x} \frac{d}{dx} \cos 3x + \cos 3x \frac{d}{dx} e^{6x}$

$$= e^{6x}(-\sin 3x)\frac{d}{dx}(3x) + \cos 3x \cdot e^{6x}\frac{d}{dx}(6x)$$

$$= e^{6x}[-3\sin 3x + 6\cos 3x]$$

$$\therefore \frac{d^2y}{dx^2} = e^{6x}\frac{d}{dx}[-3\sin 3x + 6\cos 3x] + [-3\sin 3x + 6\cos 3x]\frac{d}{dx}e^{6x}$$

$$= e^{6x}\left[-3\cos 3x\frac{d}{dx}(3x) + 6(-\sin 3x)\frac{d}{dx}(3x)\right] + [-3\sin 3x + 6\cos 3x]e^{6x}\frac{d}{dx}(6x)$$

$$= e^{6x}\{-9\cos 3x - 18\sin 3x - 18\sin 3x + 36\cos 3x\}$$

$$= e^{6x}(27\cos 3x - 36\sin 3x)$$

$$= 9e^{6x}(3\cos 3x - 4\sin 3x)$$

Q8. $\tan^{-1} x$

A.8. Let $y = \tan^{-1} x$

So, $\frac{dy}{dx} = \frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$

$$\therefore \frac{d^2y}{dx^2} = \frac{(1+x^2)\frac{d}{dx}(1) - (1)\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$= \frac{-2x}{(1+x^2)^2}$$

Q9. $\log(\log x)$

A.9. Let $y = \log(\log x)$

So, $\frac{dy}{dx}=\frac{1}{\log x}\frac{d}{dx}\log x=\frac{1}{x\log x}$

$$\therefore \frac{d^2y}{dx^2}=\frac{x\log x\frac{d}{dx}(1)-1\cdot\frac{d}{dx}(x\log x)}{(x\log x)^2}$$

$$=\frac{-\left[x\frac{d}{dx}\log x+\log x\frac{dx}{dx}\right]}{[x\log x]^2}$$

$$=\frac{-\left(x\times\frac{1}{x}+\log x\right)}{[x\log x]^2}$$

$$=\frac{-(1+\log x)}{(x\log x)^2}$$

Q10. $\sin(\log\ x)$

A.10. Let $y=\sin(\log x)$

so, $\frac{dy}{dx}=\frac{d}{dx}\sin(\log x)=\cos(\log x)\frac{d}{dx}\log x=\frac{\cos(\log x)}{x}$

$$\therefore \frac{d^2y}{dx^2}=\frac{x\frac{d}{dx}\cos(\log x)-\cos(\log x)\frac{dx}{dx}}{x^2}$$

$$=\frac{x[-\sin(\log x)]\frac{d}{dx}\log x-\cos(\log x)}{x^2}$$

$$=\frac{-\left[x\cdot\sin(\log x)\times\frac{1}{x}+\cos(\log x)\right]}{x^2}$$

$$=\frac{-[\sin(\log x)+\cos(\log x)]}{x^2}$$

Q11. If $y = 5\cos x - 3\sin x$, prove that $\frac{d^2y}{dx^2} + y = 0$

A.11. Given, $y = 5\cos x - 3\sin x$

Differentiating w r t x we get,

$$\frac{dy}{dx} = 5\frac{d}{dx}\cos x - 3\frac{d}{dx}\sin x$$

$$-5\sin x - 3\cos x.$$

Differentiating again w r t. 'x' we get,

$$\frac{d^2y}{dx^2} = -5\frac{d}{dx}\sin x - 3\frac{d}{dx}\cos x$$

$$= -5\cos x + 3\sin x$$

$$= -[5\cos x - 3\sin x]$$

$$= -y$$

$$\Rightarrow \frac{d^2y}{dx^2} + y = 0$$. Hence proved.

Q12. If $y = \cos^{-1} x$, Find $\frac{d^2y}{dx^2}$ in terms of y alone.

A.12. Given, $y = \cos^{-1} x \Rightarrow \cos y = x$

Differentiating w r t. 'x' we get,

$$\frac{dy}{dx} = \frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1-x^2}} = \frac{-1}{\sqrt{1-\cos^2 y}}$$

$$= \frac{-1}{\sqrt{\sin^2 y}}$$

$= \frac{-1}{\text{siny}}$

$-\cos ec\ y$

$\therefore \frac{d^2y}{dx^2} = -\frac{d}{dx}(\cos ec\ y)$

$= -(-\cos ec\ y \cot y)\frac{dy}{dx}$

$= \cos ec\ y \cot y(-\text{cosec}\ \ \text{y})$

$= -\cos ec^2 y \cdot \cot y$

Q13. If $y = 3\cos(\log\ x) + 4\sin(\log\ x)$, show that $x^2y_2 + xy_1 + y = 0$

A.13. Given $y = 3\cos(\log x) + 4\sin(\log x)$

So, $y_1 = \frac{dy}{dx} = 3\frac{d}{dx}\cos(\log x) + 4\frac{d}{dx}\sin(\log x)$

$y_1 = 3[-\sin(\log x)]\frac{d}{dx}\log x + 4\cos(\log x)\frac{d}{dx}(\log x)$

$y_1 = \frac{-3\sin(\log x)}{x} + \frac{4\cos(\log x)}{x}$

$\Rightarrow xy_1 = -3\sin(\log x) + 4\cos(\log x)$ ______________(1)

Differentiating eqn (1) w r t 'x' we get,

$\frac{d}{dx}(xy_1) = -3\frac{d}{dx}\sin(\log x) + 4\frac{d}{dx}\cos(\log x)$

$\Rightarrow x\frac{dy_1}{dx} + y_1\frac{dx}{dx} = -3\cos(\log(x))\frac{d}{dx}\log x + 4[-\sin(\log x)]\frac{d}{dx}\log x$

$\Rightarrow xy_2 + y_1 = \frac{-3\cos(\log x)}{x} - \frac{4\sin(\log x)}{x}$

$$\Rightarrow x^2y_2 + y_1 = -\left[3\cos(\log x) + 4\sin(\log x)\right]$$

$$\Rightarrow x^2y_2 + y_1 = -y$$

$$\Rightarrow x^2y_2 + y_1 + y = 0$$

Q14. If $y = \mathrm{A}e^{mx} + \mathrm{B}e^{nx}$, show that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$

A.14. Given, $y = Ae^{mx} + Be^{nx}$ _______(1)

So, $\frac{dy}{dx} = Ame^{mx} + Bne^{nx}$ _______(2)

$\therefore \frac{d^2y}{dx^2} = Am^2e^{mx} + Bn^2e^{nx}$ _________(3)

So, L.H.S = $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$

$$= Am^2e^{mx} + Bn^2e^{nx} - (m+n)\left[Ame^{mx} + Bne^{nx}\right] + mn\left[Ae^{mx} + Be^{nx}\right]$$

$$= Am^2e^{mx} + Bn^2e^{nx} - Am^2e^{mx} - Bmne^{nx} - Amne^{mx} - Bn^2e^{nx} + Amne^{mx} + Bmne^{nx}$$

= 0 = R.H.S.

Q15. If $y = 500e^{7x} + 600\mathrm{e}^{-7x}$, show that $\frac{d^2y}{dx^2} = 49y$

A.15. Given, $y = 500e^{7x} + 600e^{-7x}$

So, $\frac{dy}{dx} = 500 \times 7e^{7x} + 600(-7)e^{-7x}$

$$\therefore \frac{d^2y}{dx^2} = 500 \times 7^2 e^{7x} + 600 \times 7^2 e^{-7x}$$

$$=49\left[500e^{7x}+600e^{-7x}\right]$$

$$=49\times y$$

$$\Rightarrow \frac{d^2y}{dx^2}=49y$$

Q16. If $e^y(x+1)=1$, show that $\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$

A.16. Given, $e^y(x+1)=1$

$$\Rightarrow e^y=\frac{1}{x+1}$$

Taking log,

$$y\log c=\log 1-\log(x+1)$$

$$\Rightarrow y=-\log(x+1)\left\{\begin{matrix}\because \log e=1\\ \log 1=0\end{matrix}\right\}$$

Differentiating w r t 'x',

$$\frac{dy}{dx}=-\frac{1}{x+1}\frac{d}{dx}(x+1)=\frac{-1}{x+1}$$

$$\frac{d^2y}{dx^2}=-\left\{\frac{(x+1)\frac{d}{dx}(1)-(1)\frac{d}{dx}(x+1)}{(x+1)^2}\right\}$$

And

$$=\frac{1}{(x+1)^2}\quad=\left(\frac{-1}{x+1}\right)^2$$

$$\Rightarrow \frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$$

Q17. If $y=\left(\tan^{-1}x\right)^2$,show that $\left(x^2+1\right)^2 y_2+2x\left(x^2+1\right)y_1=2$

A.17. Given, $y=\left(\tan^{-1}x\right)^2$

So, $y_1=\frac{dy}{dx}=2\left(\tan^{-1}x\right)\frac{d}{dx}\tan^{-1}x$

$$\Rightarrow y_1=2\tan^{-1}x\times\frac{1}{1+x^2}$$

$$\left(x^2+1\right)y_1=2\tan^{-1}x$$

Differentiating again w r t ‘x’ we get,

$$\left(x^2+1\right)\frac{dy_1}{dx}+y_1\frac{d}{dx}\left(x^2+1\right)=2\frac{d}{dx}\tan^{-1}x$$

$$\Rightarrow \left(x^2+1\right)y_2+y_1\left(2x\right)=\frac{\quad}{1+x^2}$$

$$\Rightarrow \left(x^2+1\right)^2 y_2+2x\left(1+x^2\right)y_1=2$$

Hence proved.