Chapter 5 : Continuity and Differentiability

Exercise 5.6

If x and y are connected parametrically by the equations given in Exercise 1 to 10, without eliminating the parameter, find dy/dx.

Q1. $x = 2at^2, y = at^4$

A.1. Given, x = 2at2 and y = at4. Differentiation w r t we get,

$\frac{dx}{dt} = 4at.$ and $\frac{dy}{dt} = 4at^3.$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4at^3}{4at} = t^2.$$

Q2. $x = a \cos\theta,\ y = b \cos\theta$

A.2. Given, x = a cos θ and y = b cos θ

Differentiating w r t. θ we get,

$$\frac{dx}{d\theta} = a\frac{d}{d\theta}\cos\theta \quad \& \quad \frac{dy}{d\theta} = b\frac{d}{d\theta}\cos\theta.$$

= –a sin θ   = –b sin θ.

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-b\sin\theta}{-a\sin\theta} = \frac{b}{a}.$$

Q3. $x = \sin t, y = \cos 2t$

A.3. Given, x = sin t and y = cos2t. differentiation w r t. 't' we get,

$$\frac{dx}{dt} = \cos t \text{ and } \frac{dy}{dt} = (-\sin 2t)\frac{d\,2t}{dt}$$

$$= -2\sin 2t$$

$$= -2(2\sin t\cos t)$$

$$= -4\sin t \cos t$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{-4\sin t\cos t}{\cos t}$$

= –4 sin t

Q4. $x = 4t, y = \frac{4}{t}$

A.4. Given, x = 4t and y = $\frac{4}{t}$ Differentiating w r t. 't' we get,

$$\frac{dx}{dt} = 4 \quad \text{and} \quad \frac{dy}{dt} = \frac{4d(t^{-1})}{dt}$$

$= -4t^{-2}$

$\frac{-4}{t^2}$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-4/t^2}{4} = -\frac{1}{t^2}.$$

Q5. $x = \cos\theta - \cos 2\theta, y = \sin\theta - \sin 2\theta$

A.5. Given, x = cos θ – cos 2θ and y = sin θ – sin 2θ. Differentiation w r t. θ we get,

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(\cos\theta) - \frac{d}{d\theta}(\cos 2\theta)$$

$$= -\sin\theta - \frac{(-\sin 2\theta)}{0}\frac{d}{d\theta}(2\theta)$$

$$= -\sin\theta + 2\sin 2\theta$$

And $\frac{dy}{d\theta} = \frac{d}{d\theta}(\sin\theta - \sin 2\theta)$

$$= \cos\theta - \cos 2\theta \frac{d(2\theta)}{d\theta}$$

$$= \cos\theta - 2\cos 2\theta.$$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta - 2\cos 2\theta}{2\sin 2\theta - \sin\theta}.$$

Q6. $x = a(\theta - \sin\theta), y = a(1 + \cos\theta)$

A.6. Given, $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$

Differentiation w r t. θ we get,

$$\frac{dx}{d\theta}=a\frac{d}{d\theta}(\theta-\sin\theta)=a[1-\cos\theta]$$

ad $\frac{dy}{d\theta}=a\frac{d}{d\theta}(1+\cos\theta)$

$$=a(0-\sin\theta)$$

$$=-a\sin\theta$$

$$\therefore\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{-a\sin\theta}{a[1-\cos\theta]}$$

$$=\frac{-\sin\theta}{1-\cos\theta}$$

$$=-\frac{2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2}\begin{cases}\because \sin 2\theta=2\sin\theta\cos\theta\\ and\\ \cos 2\theta=1-2\sin^2\theta\end{cases}$$

$$=-\frac{\cos\theta/2}{\sin\theta/2}$$

$$=-\cot\theta/2$$

Q7. $x=\frac{\sin^3 t}{\sqrt{\cos 2t}}$, $y=\frac{\cos^3 t}{\sqrt{\cos 2t}}$

A.7. Given $x=\frac{\sin^3 t}{\sqrt{\cos 2t}}$ and $y=\frac{\cos^3 t}{\sqrt{\cos 2t}}$

Differentiating w.r.t. 't' we get,

$$\frac{dx}{dt}=\frac{\sqrt{\cos 2t}\frac{d}{dt}(\sin^3 t)-\sin^3 t\frac{d}{dt}\sqrt{\cos 2t}}{\left(\sqrt{\cos 2t}\right)^2}$$

$$= \frac{\sqrt{\cos 2t}\,3\sin^2 t\,\frac{d}{dt}(\sin t) - \sin^3 t \times \frac{1}{2\sqrt{\cos 2t}}\,\frac{d}{dt}(\cos 2t)}{\cos 2t}$$

$$= \frac{\frac{\left(\sqrt{\cos 2t}\right)^2 .\ 3\sin^2 t\cos t}{\sqrt{\cos 2t}} + \frac{\sin^3 t.\sin 2t \times 2}{2\sqrt{\cos 2t}}}{\cos 2t}$$

$$= \frac{\sin^2 t\cos t\ \ 3\cos 2t + \sin^3 t\ \ 2\sin t\cos t}{(\cos 2t)\sqrt{\cos 2t}}$$

$$= \frac{\sin^2 t\cos t\ \left(3\cos 2t + 2\sin^2 t\right)}{(\cos 2t)^{3/2}}$$

And

$$\frac{dy}{dt} = \frac{\sqrt{\cos 2t}\,\frac{d}{dt}\left(\cos^3 t\right) - \cos^3 t\,\frac{d}{dt}\sqrt{\cos 2t}}{\left(\sqrt{\cos 2t}\right)^2}$$

$$= \frac{\sqrt{\cos 2t} \times 3\cos^2 t\,\frac{d}{dt}(\cos t) - \cos^3 t \times \frac{1}{2\sqrt{\cos 2t}}\,\frac{d}{dt}(\cos 2t)}{\cos 2t}$$

$$= \frac{\sqrt{\cos 2t} \times 3\cos^2 t(-\sin t) + \cos^3 t \times \frac{1}{2\sqrt{\cos 2t}}\sin 2t.2}{\cos 2t}$$

$$= \frac{-3\cos 2t\cos^2 t\sin t + \cos^3 t \times 2\sin t\cos t}{(\cos 2t)\sqrt{\cos 2t}}$$

$$= \frac{\cos^2 t\sin t\left[2\cos^2 t - 3\cos 2t\right]}{(\cos 2t)^{3/2}}$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{\cos^2 t \sin t\left(2\cos^2 t - 3\cos 2t\right) \big/ (\cos 2t)^{3/2}}{\sin^2 t \cos t\left(3\cos 2t + 2\sin^2 t\right) \big/ (\cos 2t)^{3/2}}$$

$$= \frac{\cos t\left[2\cos^2 t - 3\left(2\cos^2 t - 1\right)\right]}{\sin t\left[3\left(1 - 2\sin^2 t\right) + 2\sin^2 t\right]} \quad \left\{\because \cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta\right.$$

$$= \frac{\cos t\left[2\cos^2 t - 6\cos^2 t + 3\right]}{\sin t\left[3 - 6\sin^2 t + 2\sin^2 t\right]}$$

$$= \frac{\cos t\left[-4\cos^2 t + 3\right]}{\sin t\left[3 - 4\sin^2 t\right]}$$

$$= \frac{-\left[4\cos^3 t - 3\cos t\right]}{\left[3\sin t - 4\sin^3 t\right]}$$

$$= -\frac{\cos 3t}{\sin 3t}$$

$$= -\cot 3t$$

Q8. $x = a\left(\cos t + \log\tan\frac{t}{2}\right) \quad y = a\sin t$

A.8. Given, $x = a\left(\cos t + \log\tan\frac{t}{2}\right)$ and $y = a\sin t$

Differentiating w r t we get,

$$\frac{dx}{dt} = a\frac{d}{dt}\left[\cos t + \log\left(\tan\frac{t}{2}\right)\right]$$

$$= a\left[-\sin t + \frac{1}{\tan t/2}\frac{d}{dt}\left(\tan\frac{t}{2}\right)\right]$$

$$= a\left[-\sin t + \frac{1}{\tan {}^{t}\!/\!_{2}}.\sec^2 \frac{t}{2}\frac{d}{dt}\left(\frac{t}{2}\right)\right]$$

$$= a\left[-\sin t + \frac{\cos {}^{t}\!/\!_{2}}{\sin {}^{t}\!/\!_{2}} \times \frac{1}{\cos^2 {}^{t}\!/\!_{2}} \times \frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{1}{2\sin {}^{t}\!/\!_{2} \cos {}^{t}\!/\!_{2}}\right]$$

$$= a\left[-\sin t + \frac{1}{\sin 2\times {}^{t}\!/\!_{2}}\right]$$

$$= a\left[-\sin t + \frac{1}{\sin t}\right] = a\left[\frac{1-\sin^2 t}{\sin t}\right]$$

$$= a\frac{\cos^2 t}{\sin t}\{\because 1 = \cos^2 x + \sin^2 x\}$$

and $\dfrac{dy}{dt} = \dfrac{d}{dt}(a\sin t) = a\cos t$

$$\therefore \frac{dy}{dx} = \frac{{}^{dy}\!/\!_{dt}}{{}^{dx}\!/\!_{dt}} = \frac{a\cos t}{a\cos^2 t/\sin t} = \frac{\sin t}{\cos t} = \tan t$$

Q9. $x = a\sec\theta, y = b\tan\theta$

A.9. Given, $x = a\sec\theta$ and $y = b\tan\theta$

Differentiating w r t θ we get,

$$\frac{dx}{d\theta} = a\frac{d}{d\theta}\sec\theta$$

$$= a\sec\theta\tan\theta$$

$$\frac{dy}{d\theta} = b\frac{d}{d\theta}\tan\theta$$

$$= b\sec^2\theta$$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b}{a}\frac{1}{\cos\theta} \times \frac{\cos\theta}{\sin\theta}$$

$$= \frac{b}{a}\frac{1}{\sin\theta} = \frac{b}{a}\operatorname{cosec}\theta.$$

Q10. $x = a(\cos\theta + \theta\sin\theta), y = a(\sin\theta - \theta\cos\theta)$

A.10. Given, $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$

Differentiating w r t. θ we get,

$$\frac{dx}{d\theta} = a\frac{d}{d\theta}[\cos\theta + \theta\sin\theta]$$

$$= a\left[-\sin\theta + \theta\frac{d}{d\theta}\sin\theta + \sin\theta\frac{d\theta}{d\theta}\right]$$

$$= a[-\sin\theta + \theta\cos\theta + \sin\theta]$$

$$= a\theta\cos\theta$$

$$\frac{dy}{d\theta} = a\frac{d}{d\theta}[\sin\theta - \theta\cos\theta]$$

$$= a\left[\cos\theta - \theta\frac{d}{d\theta}\cos\theta - \cos\theta\frac{d\theta}{d\theta}\right]$$

$$= a[\cos\theta + \theta\sin\theta - \cos\theta]$$

$$= a\theta\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

Q11. If $x=\sqrt{a^{\sin^{-1}t}}\, y=\sqrt{a^{\cos^{-1}t}}$,show that $\frac{dy}{dx}=-\frac{y}{x}$

A.11. Given, $x=\sqrt{a\sin^{-1}t}$ and $y=\sqrt{a\cos^{-1}t}$

Then, $\frac{dx}{dt}=\frac{d}{dt}\left[a^{1/2\times\sin^{-1}t}\right]\left\{\because \frac{da^x}{dx}=a^x\log a\right\}$

$=a^{\frac{1}{2}\sin^{-1}t}\log a\frac{d}{dt}\left(\frac{1}{2}\sin^{-1}t\right)$

$=\frac{1}{2}\sqrt{a^{\sin^{-1}t}}\times\log a\times\frac{1}{\sqrt{1-t^2}}$

And $\frac{dy}{dt}=\frac{d}{dt}\left(a^{\frac{1}{2}\cos^{-1}t}\right)$

$=a^{\frac{1}{2}\cos^{-1}t}\times\log a.\frac{d}{dt}\left(\frac{1}{2}\cos^{-1}t\right)$

$=\frac{1}{2}\sqrt{a^{\cos^{-1}t}}\times\log a\times\left(\frac{-1}{\sqrt{1-t^2}}\right)$

$\therefore \frac{dy}{dx}=\frac{dy/dt}{dx/dt}$

$=\frac{\frac{1}{2}\sqrt{a^{\cos^{-1}t}}\log a\left(\frac{-1}{\sqrt{1-t^2}}\right)}{\frac{1}{2}\sqrt{a^{\sin^{-1}t}}\log a\left(\frac{1}{\sqrt{1-t^2}}\right)}$

$=\frac{-\sqrt{a^{\cos^{-1}t}}}{\sqrt{a^{\sin^{-1}t}}}$

$\frac{dy}{dx}=-\frac{y}{x}$

Hence proved.