Exercise 5.4

Question 1:

Differentiate the following w.r.t. x:

$$\frac{e^x}{\sin x}$$

Answer

Let $y = \frac{e^x}{\sin x}$

By using the quotient rule, we obtain

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\sin x \frac{d}{dx}\left(e^x\right) - e^x \frac{d}{dx}\left(\sin x\right)}{\sin^2 x} \\
&= \frac{\sin x.\left(e^x\right) - e^x \cdot \left(\cos x\right)}{\sin^2 x} \\
&= \frac{e^x\left(\sin x - \cos x\right)}{\sin^2 x}, x \neq n\pi, n \in \mathbf{Z}
\end{aligned}$$

Question 2:

Differentiate the following w.r.t. x:

$$e^{\sin^{-1} x}$$

Answer

Let $y = e^{\sin^{-1} x}$

By using the chain rule, we obtain

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^{\sin^{-1}x}\right)$$

$$\Rightarrow\frac{dy}{dx}=e^{\sin^{-1}x}\cdot\frac{d}{dx}\left(\sin^{-1}x\right)$$

$$=e^{\sin^{-1}x}\cdot\frac{1}{\sqrt{1-x^2}}$$

$$=\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$$

$$\therefore\frac{dy}{dx}=\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}},x\in(-1,1)$$

Question 2:

Show that the function given by $f(x) = e^{2x}$ is strictly increasing on **R**.

Answer

Let x_1 and x_2 be any two numbers in **R**.

Then, we have:

$$x_1<x_2\Rightarrow 2x_1<2x_2\Rightarrow e^{2x_1}<e^{2x_2}\Rightarrow f(x_1)<f(x_2)$$

Hence, f is strictly increasing on **R**.

Question 3:

Differentiate the following w.r.t. x:

e^{x^3}

Answer

Let $y=e^{x^3}$

By using the chain rule, we obtain

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^{x^3}\right)=e^{x^3}\cdot\frac{d}{dx}\left(x^3\right)=e^{x^3}\cdot 3x^2=3x^2e^{x^3}$$

Question 4:

Differentiate the following w.r.t. x:

$\sin\left(\tan^{-1}e^{-x}\right)$

Answer

Let $y=\sin\left(\tan^{-1}e^{-x}\right)$

By using the chain rule, we obtain

$$\frac{dy}{dx}=\frac{d}{dx}\left[\sin\left(\tan^{-1}e^{-x}\right)\right]$$
$$=\cos\left(\tan^{-1}e^{-x}\right)\cdot\frac{d}{dx}\left(\tan^{-1}e^{-x}\right)$$
$$=\cos\left(\tan^{-1}e^{-x}\right)\cdot\frac{1}{1+\left(e^{-x}\right)^2}\cdot\frac{d}{dx}\left(e^{-x}\right)$$
$$=\frac{\cos\left(\tan^{-1}e^{-x}\right)}{1+e^{-2x}}\cdot e^{-x}\cdot\frac{d}{dx}(-x)$$
$$=\frac{e^{-x}\cos\left(\tan^{-1}e^{-x}\right)}{1+e^{-2x}}\times(-1)$$
$$=\frac{-e^{-x}\cos\left(\tan^{-1}e^{-x}\right)}{1+e^{-2x}}$$

Question 5:

Differentiate the following w.r.t. *x*:

$\log\left(\cos e^{x}\right)$

Answer

Let $y=\log\left(\cos e^{x}\right)$

By using the chain rule, we obtain

$$\frac{dy}{dx}=\frac{d}{dx}\left[\log\left(\cos e^{x}\right)\right]$$
$$=\frac{1}{\cos e^{x}}\cdot\frac{d}{dx}\left(\cos e^{x}\right)$$
$$=\frac{1}{\cos e^{x}}\cdot\left(-\sin e^{x}\right)\cdot\frac{d}{dx}\left(e^{x}\right)$$
$$=\frac{-\sin e^{x}}{\cos e^{x}}\cdot e^{x}$$
$$=-e^{x}\tan e^{x},e^{x}\neq(2n+1)\frac{\pi}{2},n\in\mathbf{N}$$

Question 6:

Differentiate the following w.r.t. *x*:

$e^x + e^{x^2} + \ldots + e^{x^5}$

Answer

$$\frac{d}{dx}\left(e^x + e^{x^2} + \ldots + e^{x^5}\right)$$
$$= \frac{d}{dx}\left(e^x\right) + \frac{d}{dx}\left(e^{x^2}\right) + \frac{d}{dx}\left(e^{x^3}\right) + \frac{d}{dx}\left(e^{x^4}\right) + \frac{d}{dx}\left(e^{x^5}\right)$$
$$= e^x + \left[e^{x^2} \times \frac{d}{dx}\left(x^2\right)\right] + \left[e^{x^3} \cdot \frac{d}{dx}\left(x^3\right)\right] + \left[e^{x^4} \cdot \frac{d}{dx}\left(x^4\right)\right] + \left[e^{x^5} \cdot \frac{d}{dx}\left(x^5\right)\right]$$
$$= e^x + \left(e^{x^2} \times 2x\right) + \left(e^{x^3} \times 3x^2\right) + \left(e^{x^4} \times 4x^3\right) + \left(e^{x^5} \times 5x^4\right)$$
$$= e^x + 2xe^{x^2} + 3x^2 e^{x^3} + 4x^3 e^{x^4} + 5x^4 e^{x^5}$$

Question 7:

Differentiate the following w.r.t. *x*:

$\sqrt{e^{\sqrt{x}}}, x > 0$

Answer

Let $y = \sqrt{e^{\sqrt{x}}}$

Then, $y^2 = e^{\sqrt{x}}$

By differentiating this relationship with respect to *x*, we obtain

$y^2 = e^{\sqrt{x}}$

$\Rightarrow 2y\frac{dy}{dx} = e^{\sqrt{x}}\frac{d}{dx}\left(\sqrt{x}\right)$ [By applying the chain rule]

$\Rightarrow 2y\frac{dy}{dx} = e^{\sqrt{x}}\frac{1}{2}\cdot\frac{1}{\sqrt{x}}$

$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4y\sqrt{x}}$

$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4\sqrt{e^{\sqrt{x}}}\sqrt{x}}$

$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4\sqrt{xe^{\sqrt{x}}}}, \ x > 0$

Question 8:

Differentiate the following w.r.t. *x*:

$\log(\log x), x > 1$

Answer

Let $y = \log(\log x)$

By using the chain rule, we obtain

$\frac{dy}{dx} = \frac{d}{dx}\left[\log(\log x)\right]$

$= \frac{1}{\log x}\cdot\frac{d}{dx}(\log x)$

$= \frac{1}{\log x}\cdot\frac{1}{x}$

$= \frac{1}{x\log x}$, $x > 1$

Question 9:

Differentiate the following w.r.t. *x*:

$\frac{\cos x}{\log x}, x > 0$

Answer

Let $y = \dfrac{\cos x}{\log x}$

By using the quotient rule, we obtain

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(\cos x) \times \log x - \cos x \times \frac{d}{dx}(\log x)}{(\log x)^2}$$

$$= \frac{-\sin x \log x - \cos x \times \frac{1}{x}}{(\log x)^2}$$

$$= \frac{-[x \log x . \sin x + \cos x]}{x(\log x)^2}, x > 0$$

Question 10:

Differentiate the following w.r.t. *x*:

$\cos(\log x + e^x), x > 0$

Answer

Let $y = \cos(\log x + e^x)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = -\sin(\log x + e^x) \cdot \frac{d}{dx}(\log x + e^x)$$

$$= -\sin(\log x + e^x) \cdot \left[\frac{d}{dx}(\log x) + \frac{d}{dx}(e^x)\right]$$

$$= -\sin(\log x + e^x) \cdot \left(\frac{1}{x} + e^x\right)$$

$$= -\left(\frac{1}{x} + e^x\right) \sin(\log x + e^x), x > 0$$