

## Exercise 5.2

**Question 1:**

Differentiate the functions with respect to  $x$ .

$$\sin(x^2 + 5)$$

Answer

$$\text{Let } f(x) = \sin(x^2 + 5), \quad u(x) = x^2 + 5, \text{ and } v(t) = \sin t$$

$$\text{Then, } (f \circ u)(x) = v(u(x)) = v(x^2 + 5) = \sin(x^2 + 5) = f(x)$$

Thus,  $f$  is a composite of two functions.

$$\text{Put } t = u(x) = x^2 + 5$$

Then, we obtain

$$\frac{dv}{dt} = \frac{d}{dt}(\sin t) = \cos t = \cos(x^2 + 5)$$

$$\frac{dt}{dx} = \frac{d}{dx}(x^2 + 5) = \frac{d}{dx}(x^2) + \frac{d}{dx}(5) = 2x + 0 = 2x$$

$$\text{Therefore, by chain rule, } \frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos(x^2 + 5) \times 2x = 2x \cos(x^2 + 5)$$

**Alternate method**

$$\begin{aligned} \frac{d}{dx}[\sin(x^2 + 5)] &= \cos(x^2 + 5) \cdot \frac{d}{dx}(x^2 + 5) \\ &= \cos(x^2 + 5) \cdot \left[ \frac{d}{dx}(x^2) + \frac{d}{dx}(5) \right] \\ &= \cos(x^2 + 5) \cdot [2x + 0] \\ &= 2x \cos(x^2 + 5) \end{aligned}$$

**Question 2:**

Differentiate the functions with respect to  $x$ .

$$\cos(\sin x)$$

Answer

Let  $f(x) = \cos(\sin x)$ ,  $u(x) = \sin x$ , and  $v(t) = \cos t$

Then,  $(v \circ u)(x) = v(u(x)) = v(\sin x) = \cos(\sin x) = f(x)$

Thus,  $f$  is a composite function of two functions.

Put  $t = u(x) = \sin x$

$$\therefore \frac{dv}{dt} = \frac{d}{dt}[\cos t] = -\sin t = -\sin(\sin x)$$

$$\frac{dt}{dx} = \frac{d}{dx}(\sin x) = \cos x$$

By chain rule,  $\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = -\sin(\sin x) \cdot \cos x = -\cos x \sin(\sin x)$

**Alternate method**

$$\frac{d}{dx}[\cos(\sin x)] = -\sin(\sin x) \cdot \frac{d}{dx}(\sin x) = -\sin(\sin x) \cdot \cos x = -\cos x \sin(\sin x)$$

**Question 3:**

Differentiate the functions with respect to  $x$ .

$$\sin(ax + b)$$

Answer

Let  $f(x) = \sin(ax + b)$ ,  $u(x) = ax + b$ , and  $v(t) = \sin t$

Then,  $(v \circ u)(x) = v(u(x)) = v(ax + b) = \sin(ax + b) = f(x)$

Thus,  $f$  is a composite function of two functions,  $u$  and  $v$ .

Put  $t = u(x) = ax + b$

Therefore,

$$\frac{dv}{dt} = \frac{d}{dt}(\sin t) = \cos t = \cos(ax + b)$$

$$\frac{dt}{dx} = \frac{d}{dx}(ax + b) = \frac{d}{dx}(ax) + \frac{d}{dx}(b) = a + 0 = a$$

Hence, by chain rule, we obtain

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos(ax + b) \cdot a = a \cos(ax + b)$$

**Alternate method**

$$\begin{aligned}
 \frac{d}{dx} [\sin(ax+b)] &= \cos(ax+b) \cdot \frac{d}{dx}(ax+b) \\
 &= \cos(ax+b) \cdot \left[ \frac{d}{dx}(ax) + \frac{d}{dx}(b) \right] \\
 &= \cos(ax+b) \cdot (a+0) \\
 &= a \cos(ax+b)
 \end{aligned}$$

**Question 4:**

Differentiate the functions with respect to  $x$ .

$$\sec(\tan(\sqrt{x}))$$

Answer

Let  $f(x) = \sec(\tan \sqrt{x})$ ,  $u(x) = \sqrt{x}$ ,  $v(t) = \tan t$ , and  $w(s) = \sec s$

Then,  $(w \circ v \circ u)(x) = w[v(u(x))] = w[v(\sqrt{x})] = w(\tan \sqrt{x}) = \sec(\tan \sqrt{x}) = f(x)$

Thus,  $f$  is a composite function of three functions,  $u$ ,  $v$ , and  $w$ .

Put  $s = v(t) = \tan t$  and  $t = u(x) = \sqrt{x}$

$$\begin{aligned}
 \text{Then, } \frac{dw}{ds} &= \frac{d}{ds}(\sec s) = \sec s \tan s = \sec(\tan t) \cdot \tan(\tan t) & [s = \tan t] \\
 &= \sec(\tan \sqrt{x}) \cdot \tan(\tan \sqrt{x}) & [t = \sqrt{x}]
 \end{aligned}$$

$$\begin{aligned}
 \frac{ds}{dt} &= \frac{d}{dt}(\tan t) = \sec^2 t = \sec^2 \sqrt{x} \\
 \frac{dt}{dx} &= \frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2} \cdot x^{\frac{1}{2}-1} = \frac{1}{2\sqrt{x}}
 \end{aligned}$$

Hence, by chain rule, we obtain

$$\begin{aligned}
 \frac{dt}{dx} &= \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx} \\
 &= \sec(\tan \sqrt{x}) \cdot \tan(\tan \sqrt{x}) \times \sec^2 \sqrt{x} \times \frac{1}{2\sqrt{x}} \\
 &= \frac{1}{2\sqrt{x}} \sec^2 \sqrt{x} \sec(\tan \sqrt{x}) \tan(\tan \sqrt{x}) \\
 &= \frac{\sec^2 \sqrt{x} \sec(\tan \sqrt{x}) \tan(\tan \sqrt{x})}{2\sqrt{x}}
 \end{aligned}$$

**Alternate method**

$$\begin{aligned}
 \frac{d}{dx} [\sec(\tan \sqrt{x})] &= \sec(\tan \sqrt{x}) \cdot \tan(\tan \sqrt{x}) \cdot \frac{d}{dx} (\tan \sqrt{x}) \\
 &= \sec(\tan \sqrt{x}) \cdot \tan(\tan \sqrt{x}) \cdot \sec^2(\sqrt{x}) \cdot \frac{d}{dx} (\sqrt{x}) \\
 &= \sec(\tan \sqrt{x}) \cdot \tan(\tan \sqrt{x}) \cdot \sec^2(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} \\
 &= \frac{\sec(\tan \sqrt{x}) \cdot \tan(\tan \sqrt{x}) \sec^2(\sqrt{x})}{2\sqrt{x}}
 \end{aligned}$$

**Question 5:**

Differentiate the functions with respect to  $x$ .

$$\frac{\sin(ax+b)}{\cos(cx+d)}$$

Answer

The given function is  $f(x) = \frac{\sin(ax+b)}{\cos(cx+d)} = \frac{g(x)}{h(x)}$ , where  $g(x) = \sin(ax+b)$  and  $h(x) = \cos(cx+d)$

$$\therefore f' = \frac{g'h - gh'}{h^2}$$

Consider  $g(x) = \sin(ax+b)$

Let  $u(x) = ax+b, v(t) = \sin t$

Then,  $(v \circ u)(x) = v(u(x)) = v(ax+b) = \sin(ax+b) = g(x)$

$\therefore g$  is a composite function of two functions,  $u$  and  $v$ .

$$\text{Put } t = u(x) = ax + b$$

$$\frac{dv}{dt} = \frac{d}{dt}(\sin t) = \cos t = \cos(ax + b)$$

$$\frac{dt}{dx} = \frac{d}{dx}(ax + b) = \frac{d}{dx}(ax) + \frac{d}{dx}(b) = a + 0 = a$$

Therefore, by chain rule, we obtain

$$g' = \frac{dg}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos(ax + b) \cdot a = a \cos(ax + b)$$

$$\text{Consider } h(x) = \cos(cx + d)$$

$$\text{Let } p(x) = cx + d, \quad q(y) = \cos y$$

$$\text{Then, } (q \circ p)(x) = q(p(x)) = q(cx + d) = \cos(cx + d) = h(x)$$

$\therefore h$  is a composite function of two functions,  $p$  and  $q$ .

$$\text{Put } y = p(x) = cx + d$$

$$\frac{dq}{dy} = \frac{d}{dy}(\cos y) = -\sin y = -\sin(cx + d)$$

$$\frac{dy}{dx} = \frac{d}{dx}(cx + d) = \frac{d}{dx}(cx) + \frac{d}{dx}(d) = c$$

Therefore, by chain rule, we obtain

$$h' = \frac{dh}{dx} = \frac{dq}{dy} \cdot \frac{dy}{dx} = -\sin(cx + d) \times c = -c \sin(cx + d)$$

$$\begin{aligned}
 \therefore f' &= \frac{a \cos(ax+b) \cdot \cos(cx+d) - \sin(ax+b) \{-c \sin(cx+d)\}}{[\cos(cx+d)]^2} \\
 &= \frac{a \cos(ax+b)}{\cos(cx+d)} + c \sin(ax+b) \cdot \frac{\sin(cx+d)}{\cos(cx+d)} \times \frac{1}{\cos(cx+d)} \\
 &= a \cos(ax+b) \sec(cx+d) + c \sin(ax+b) \tan(cx+d) \sec(cx+d)
 \end{aligned}$$

**Question 6:**

Differentiate the functions with respect to  $x$ .

$$\cos x^3 \cdot \sin^2(x^5)$$

Answer

The given function is  $\cos x^3 \cdot \sin^2(x^5)$

$$\begin{aligned}
 \frac{d}{dx} [\cos x^3 \cdot \sin^2(x^5)] &= \sin^2(x^5) \times \frac{d}{dx} (\cos x^3) + \cos x^3 \times \frac{d}{dx} [\sin^2(x^5)] \\
 &= \sin^2(x^5) \times (-\sin x^3) \times \frac{d}{dx} (x^3) + \cos x^3 \times 2 \sin(x^5) \cdot \frac{d}{dx} [\sin x^5] \\
 &= -\sin x^3 \sin^2(x^5) \times 3x^2 + 2 \sin x^5 \cos x^3 \cdot \cos x^5 \times \frac{d}{dx} (x^5) \\
 &= -3x^2 \sin x^3 \cdot \sin^2(x^5) + 2 \sin x^5 \cos x^5 \cos x^3 \cdot 5x^4 \\
 &= 10x^4 \sin x^5 \cos x^5 \cos x^3 - 3x^2 \sin x^3 \sin^2(x^5)
 \end{aligned}$$

**Question 7:**

Differentiate the functions with respect to  $x$ .

$$2\sqrt{\cot(x^2)}$$

Answer

$$\begin{aligned}
 & \frac{d}{dx} \left[ 2\sqrt{\cot(x^2)} \right] \\
 &= 2 \cdot \frac{1}{2\sqrt{\cot(x^2)}} \times \frac{d}{dx} [\cot(x^2)] \\
 &= \frac{\sin(x^2)}{\cos(x^2)} \times -\operatorname{cosec}^2(x^2) \times \frac{d}{dx}(x^2) \\
 &= -\frac{\sin(x^2)}{\cos(x^2)} \times \frac{1}{\sin^2(x^2)} \times (2x) \\
 &= \frac{-2x}{\sqrt{\cos x^2} \sqrt{\sin x^2} \sin x^2} \\
 &= \frac{-2\sqrt{2}x}{\sqrt{2\sin x^2 \cos x^2} \sin x^2} \\
 &= \frac{-2\sqrt{2}x}{\sin x^2 \sqrt{\sin 2x^2}}
 \end{aligned}$$

**Question 8:**

Differentiate the functions with respect to  $x$ .

$$\cos(\sqrt{x})$$

Answer

$$\text{Let } f(x) = \cos(\sqrt{x})$$

$$\text{Also, let } u(x) = \sqrt{x}$$

$$\text{And, } v(t) = \cos t$$

$$\text{Then, } (v \circ u)(x) = v(u(x))$$

$$= v(\sqrt{x})$$

$$= \cos \sqrt{x}$$

$$= f(x)$$

Clearly,  $f$  is a composite function of two functions,  $u$  and  $v$ , such that

$$t = u(x) = \sqrt{x}$$



$$\begin{aligned}\text{Then, } \frac{dt}{dx} &= \frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{-\frac{1}{2}} \\ &= \frac{1}{2\sqrt{x}}\end{aligned}$$

$$\begin{aligned}\text{And, } \frac{dv}{dt} &= \frac{d}{dt}(\cos t) = -\sin t \\ &= -\sin(\sqrt{x})\end{aligned}$$

By using chain rule, we obtain

$$\begin{aligned}\frac{dv}{dx} &= \frac{dv}{dt} \cdot \frac{dt}{dx} \\ &= -\sin(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} \\ &= -\frac{1}{2\sqrt{x}} \sin(\sqrt{x}) \\ &= -\frac{\sin(\sqrt{x})}{2\sqrt{x}}\end{aligned}$$

**Alternate method**

$$\begin{aligned}\frac{d}{dx}[\cos(\sqrt{x})] &= -\sin(\sqrt{x}) \cdot \frac{d}{dx}(\sqrt{x}) \\ &= -\sin(\sqrt{x}) \times \frac{d}{dx}\left(x^{\frac{1}{2}}\right) \\ &= -\sin \sqrt{x} \times \frac{1}{2}x^{-\frac{1}{2}} \\ &= \frac{-\sin \sqrt{x}}{2\sqrt{x}}\end{aligned}$$

**Question 9:**

Prove that the function  $f$  given by

$$f(x) = |x - 1|, \quad x \in \mathbf{R} \text{ is not differentiable at } x = 1.$$

**Answer**

The given function is  $f(x) = |x - 1|, \quad x \in \mathbf{R}$



It is known that a function  $f$  is differentiable at a point  $x = c$  in its domain if both

$$\lim_{h \rightarrow 0^-} \frac{f(c+h) - f(c)}{h} \text{ and } \lim_{h \rightarrow 0^+} \frac{f(c+h) - f(c)}{h} \text{ are finite and equal.}$$

To check the differentiability of the given function at  $x = 1$ ,

consider the left hand limit of  $f$  at  $x = 1$

$$\begin{aligned} \lim_{h \rightarrow 0^-} \frac{f(1+h) - f(1)}{h} &= \lim_{h \rightarrow 0^-} \frac{|1+h-1| - |1-1|}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{|h| - 0}{h} = \lim_{h \rightarrow 0^-} \frac{-h}{h} \quad (h < 0 \Rightarrow |h| = -h) \\ &= -1 \end{aligned}$$

Consider the right hand limit of  $f$  at  $x = 1$

$$\begin{aligned} \lim_{h \rightarrow 0^+} \frac{f(1+h) - f(1)}{h} &= \lim_{h \rightarrow 0^+} \frac{|1+h-1| - |1-1|}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{|h| - 0}{h} = \lim_{h \rightarrow 0^+} \frac{h}{h} \quad (h > 0 \Rightarrow |h| = h) \\ &= 1 \end{aligned}$$

Since the left and right hand limits of  $f$  at  $x = 1$  are not equal,  $f$  is not differentiable at  $x = 1$

### Question 10:

Prove that the greatest integer function defined by  $f(x) = [x], 0 < x < 3$  is not differentiable at  $x = 1$  and  $x = 2$ .

**Answer**

The given function  $f$  is  $f(x) = [x], 0 < x < 3$

It is known that a function  $f$  is differentiable at a point  $x = c$  in its domain if both

$$\lim_{h \rightarrow 0^-} \frac{f(c+h) - f(c)}{h} \text{ and } \lim_{h \rightarrow 0^+} \frac{f(c+h) - f(c)}{h} \text{ are finite and equal.}$$

To check the differentiability of the given function at  $x = 1$ , consider the left hand limit of  $f$  at  $x = 1$

$$\begin{aligned}\lim_{h \rightarrow 0^-} \frac{f(1+h) - f(1)}{h} &= \lim_{h \rightarrow 0^-} \frac{[1+h] - [1]}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{0-1}{h} = \lim_{h \rightarrow 0^-} \frac{-1}{h} = \infty\end{aligned}$$

Consider the right hand limit of  $f$  at  $x = 1$

$$\begin{aligned}\lim_{h \rightarrow 0^+} \frac{f(1+h) - f(1)}{h} &= \lim_{h \rightarrow 0^+} \frac{[1+h] - [1]}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{1-1}{h} = \lim_{h \rightarrow 0^+} 0 = 0\end{aligned}$$

Since the left and right hand limits of  $f$  at  $x = 1$  are not equal,  $f$  is not differentiable at  $x = 1$

To check the differentiability of the given function at  $x = 2$ , consider the left hand limit of  $f$  at  $x = 2$

$$\begin{aligned}\lim_{h \rightarrow 0^-} \frac{f(2+h) - f(2)}{h} &= \lim_{h \rightarrow 0^-} \frac{[2+h] - [2]}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{1-2}{h} = \lim_{h \rightarrow 0^-} \frac{-1}{h} = \infty\end{aligned}$$

Consider the right hand limit of  $f$  at  $x = 2$

$$\begin{aligned}\lim_{h \rightarrow 0^+} \frac{f(2+h) - f(2)}{h} &= \lim_{h \rightarrow 0^+} \frac{[2+h] - [2]}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{2-2}{h} = \lim_{h \rightarrow 0^+} 0 = 0\end{aligned}$$

Since the left and right hand limits of  $f$  at  $x = 2$  are not equal,  $f$  is not differentiable at  $x = 2$

