Chapter 5: CONTINUITY AND DIFFERENTIABILITY

Exercise – 5.1

Q1. Prove that the function $f(x) = 5x - 3f$ is continuous at $x = 0$, at $x = -3$ and at $x = 5$.

A.1. Given, $f(x) = 5x - 3$

At $x = 0$, $\lim_{x\to 0} f(x) = \lim_{x\to 0} 5x - 3 = 5 \times 0 - 3 = -3.$

So f is continuous at $x = 1$.

At $x = -3$, $\lim_{x\to -3} f(x) = \lim_{x\to -3} 5x - 3 = 5(-3) - 3 = -15 - 3$

$= -18.$

So f is continuous at $x = -3$.

At $x = 5$, $\lim_{x\to 5} f(x) = \lim_{x\to 5} .5x - 3 = 5.5 - 3 = 25 - 3 = 22.$

So, f is continuous at $x = 5$.

Q2. Examine the continuity of the function $f(x) = 2x^2 - 1$ at x = 3.

A.2. Given, $f(x) = 2x^2 - 1$

At $x = 3$

Lim $f(x) = \lim_{x\to 3} f(x) = \lim_{x\to} 2(3)^2 - 1 = 18 - 1 = 17.$

So, f is continuous at $x = 3$.

Q3. Examine the following functions for continuity.

(a) $f(x) = x - 5$ (b) $f(x) = \dfrac{1}{x-5}, x \neq 5$

(c) $f(x) = \dfrac{x^2-25}{x+5}, x \neq -5$ (d) $f(x) = |x-5|$

A.3. (*a*) Given, $f(x) = x - 5$.

The given $f\,x^{n}$ is *a* polynernial $f\,x^{n}$ and as every pohyouraial $f\,x^{n}$ is continuous in its domain R we conclude that $f(x)$ is continuous.

(*b*). Given, $f(x) = \dfrac{1}{x-5}, x \neq 5$

For any $a \in \mathbb{R} - \{5\}$,

$$\lim_{x\to a} f(x) = \lim_{x\to a} \frac{1}{(x-5)} = \frac{1}{a-5}.$$

and $f(a) = \dfrac{1}{a-5}$

i e, $\underset{x\to a}{\text{him}}\ f(x) = f(a)$.

Hence f is continuous in its domain.

(c) Given, $f(x) = \dfrac{x^2-25}{x+5}, \quad x \neq -5$

For any $a \in \mathbb{R} - \{-5\}$

$$\lim_{x\to a} f(x) = \lim_{x\to a} \frac{x^2-25}{x+5} = \frac{a^2-25}{a+5} = \frac{(a-5)(a+5)}{a+5} = a - 5$$

And $f(a) = \dfrac{a^2-25}{a+5} = \dfrac{(a-5)(a+5)}{a+5}.$

$= a - 5$

$\therefore \lim_{x\to a} f(x) = f(a)$.

So, f is continuous in its domain.

(d) Given $f(a) = |x-5| = \begin{cases} x-5 & \text{, if } x-5>0 \Rightarrow x \geq 5 \\ -(x-5) & \text{if } x-5<0 \Rightarrow x<5. \end{cases}$

For $x = c < 5$.

$f(c) = -(c-5) = 5-c$.

$\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} -(x-5) = -(c-5) = 5-c$.

$\therefore f(c) = \lim\limits_{x \to c} f(x)$.

So f is continuous.

For $x = c > 5$.

$f(c) = (x-5) = c-5$

$\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} (x-5) = c-5$.

$\therefore f(c) = \lim\limits_{x \to c} f(x)$

So, f is continuous.

For $x = c = 5$,

$f(5) = 5-5 = 0$

$\lim\limits_{x \to 5} f(a) = \lim\limits_{x \to 5} -(x-5) = -(5-5) = 0$

$\lim\limits_{x \to 5^+} f(x) = \lim\limits_{x \to 5^+} (x-5) = 5-5 = 0$

$\therefore \underset{x \to 5^+}{\text{ten}} + (x) = \underset{x \to c^+}{\text{sin}} + (x) = f(\text{c})$

Hence f is continuous.

Q4. **Prove that the function $f(x) = x^n$ continuous at $x = n$, where n is a positive integer.**

A.4. Given, $f(x) = x^{n} > n$ = positive.

At $x = 2$,

$(x) = n^{n}$.

$\lim_{x \to n} f(x) = \lim_{x \to n} x^{n} = n^{n}$

$\therefore \lim_{x \to n} f(x) = f(x)$

So f is continuous at $x = n$.

Q5. **Is the function f defined by**

$$f(x) = \begin{cases} x, \text{if } x \le 1 \\ 5, \text{if } x > 1 \end{cases}$$

continuous at $x = 0$? At $x = 2$?

Find all points of discontinuity of f, where f is defined by

A.5. Given, $f(a) = \begin{cases} x, \text{ if } x \le 1 \\ 5, \text{ if } x > 1. \end{cases}$

At $x = 0$,

$(0) = 0$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} x = 0$

$\therefore \lim_{x \to 0} f(x) = f(0)$

So, f is continuous at $x = 0$.

At $x = 1$,

Left hand limit,

L.H.L = $\lim_{x \to 1} f(x) = \lim_{x \to 1} x = 1$.

Right hand limit,

R. H. L. = $\lim_{x \to 1^{+}} f(x) = \lim_{x \to 1^{+}} 5 = 5$.

$\therefore$ L.H.L. $\neq$ R.H.L.

So, f is not continuous at $x = 1$.

At $x = 2$,

$f(2) = 5$.

$\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 2} 5 = 5$

$\therefore \lim\limits_{\lim \to 2} f\ (x) = f\,(2)$

So f is continuous at $x = 2$.

Find all points of discontinuity of f, where f is defined by

Q6. $f(x) = \begin{cases} 2x+3 \text{ if } x \le 2 \\ 2x-3 \text{ if } x > 2. \end{cases}$

A.6. Given $f(x) = \begin{cases} 2x+3 \text{ if } x \le 2 \\ 2x-3 \text{ if } x > 2. \end{cases}$

For $x = c < 2$,

F $(c) = 2c + 3$

$\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} 2x + 3 = 2c + 3$

$\therefore \lim\limits_{x \to c} f\,(x) = f(c)$

So f is continuous at $x\,|<|\,2$.

For $x = c > 2$.

F $(c) = 2c - 3$

$\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} 2x - 3 = 2c - 3$

$\therefore \lim\limits_{x \to c} f(x) = f(c)$

So f is continuous at $x|>|\ 2$.

For $x = c = 2$,

L.H.L. = $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} .2x + 3 = 2.\ 2 + 3 = 4 + 3 = 7$.

R.H.L. = $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} \quad 2x - 3 = 2.\ 2 - 3 = 4 - 3 = 1$.

$\therefore$ L·H·L· $\neq$ R·H·L·

$\therefore f$ is not continuous at $x = 2$.i e, point of discontinuity

Q7. $f(x) = \begin{cases} |x| + 3 \text{ if } x \le -3 \\ -2x \text{ if } -3 < x < 3 \\ 6x + 2 \text{ if } x \ge 3 \end{cases}$

A.7. Given, $f(x) = \begin{cases} |x| + 3 \text{ if } x \le -3 \\ -2x \text{ if } -3 < x < 3 \\ 6x + 2 \text{ if } x \ge 3 \end{cases}$

For $x = \ c < -3$,

$f(-3) = -e + 3 \qquad (\therefore x < -3, |x| = -x)$

$\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} |x| + 3 = -a + 3$.

$\therefore \lim\limits_{x\to c} f(x) = f(c)$

So, f is continuous at $x = c < -3$.

For $x = c > 3$

$f(3) = 6.3 + 2 = 18 + 2 = 20$

$\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} 6x + 2 = 18 + 2 = 20$

$\therefore \lim\limits_{x\to c} f(x) = f(c)$. So f is continuous at $x = c > 3$.

For. C = – 3,

$f(-3) = -(-3) + 3 = 6.$

$\lim_{x\to c^-} f(x) = \lim_{x\to c^-} .-x + 3 = -(-3) + 3 = 6.$

$\lim_{x\to c^+} f(x) = \lim_{x\to c^+} (-2x) = -2(-3) = 6.$

$\therefore \lim_{x\to c^-} f(x) = \lim_{x\to c^-} f(x) = f(-3)$

So, f is continuous at $x = \underline{c} = -3$.

For $c = 3$,

$f(3) = 6.3 + 2 = 18 + = 20.$

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} -2x = -2(3) = -6$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (6x + 2) = 6.3 + 2 = 20$

$\therefore \lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^-} f(x).$

f is not continuous at $x = 3$ point of discontinuity

Q8. $f(x) = \begin{cases} \frac{|x|}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$

A.8. Given, $f(x) = f(x) = |x| - |x+1|$.

For $x = c < 0$,

$f(c) = -1$

$\lim_{x\to 0} f(x) = \lim_{x\to 0} -1 = -1$

$\therefore f(c) = \lim_{x\to 0} f(x)$

f is continuous at $x|<|\ 0$.

For $x = c > 0$,

F $(c) = 1$

$\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} \neq = 1.$

$\therefore f(c) = \lim\limits_{x \to c} f(x)$

f is continuous at $x > 0$.

For $x = c - 0$.

L.H.L. = $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (-1) = -1$

R.H.L. $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} 1 = 1$

$\therefore$ L.H.L. $\neq$ R.H.L.

is now continuous at $x = 0$, π point of discontinuity of f is at $x = 0$.

Q9. $f(x) = \begin{cases} \dfrac{x}{|x|}, \text{if } x < 0 \\ -1, \text{ if } x \geq 0 \end{cases}$

A.9. Given, $f(x) = \begin{cases} \dfrac{x}{|x|} \text{ if } x < 0 \\ -1 \text{ if } x \geq 0 \end{cases} = \begin{cases} \dfrac{x}{-x} = -1 \text{ if } x < 0 \\ \dfrac{x}{2} - 1 \text{ if } x \geq 0. \end{cases}$

So, $f(x) = -1 \ \forall x \in$ R.

Hence, $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} -1 = -1 \ \forall \in \in \mathbb{R}$

$\therefore f$ is continuous in its domain

Hence, f has no point of discontinuity.

Q10. $f(x) = \begin{cases} x+1, \text{if } \ x \geq 1 \\ x^2+1, \text{ if } \ x < 1 \end{cases}$

A.10. Given, $f(x) = \begin{cases} x+1, \pi\ x \geq 1 \\ x^2+1,\ \pi\ x < 1. \end{cases}$

For $x = c < 1$,

$\lim_{x \to c} f(x) = \lim_{x \to c} x^2 + 1 = c^2 + 1$

$\therefore \lim_{x \to c} f(x) = f(c)$

So f is continuous at $x = c < 1$.

For $x = c > 1$,

F $(c) = c + 1$

$\lim_{x \to c} f(x) = \lim_{x \to c} x + 1 = c + 1$

$\therefore \lim_{x \to c} f(x) = f(c)$

So, f is continuous at $x = c > 1$.

For $x = c = 1$, + (1) = 1 + 1 = 2

L.H.L. = $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 + 1 = 1^2 + 1 = 2.$

R.H.L. = $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x + 1 = .1 + 1 = 2$

$\therefore$ L.H.L = R.H.L. = $f(1)$

So, f is continuous at $x = 1$.Hence f has no point of discontinuity.

Q11. $f(x) = \begin{cases} x^3 - 3, & \text{if } x \leq 2 \\ x^2 + 1, & \text{if } x > 2 \end{cases}$

A.11. Given $f(x) = \begin{cases} x^3 - 3 & \text{if } x|\leqslant|2 \\ x^2 + 1 & \text{if } x > 2 \end{cases}$

For $x = c < 2$,

$f(c) = c^3 - 3$

$\lim_{x \to c} f(x) = \lim_{x \to c} x^3 - 3 = c^3 - 3.$

So f is continuous at $x|<|$ 2.

For $x = c > 2$

$f(c) = x^2 + 1 = c^2 + 1$

$\lim_{x \to 2} f(x) = \lim_{x \to 2} x^2 + 1 = c^2 + 1 = f(c)$

So, f is continuous at $x|>|$ 2.

For $x = c = 2$, $f(2) = 2^3 - 3 = 8 - 3 = 5.$

L.H.L. $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x^3 - 3 = 2^3 - 3 = 5.$

R.H.L. $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x^2 + 1 = 2^2 + 1 = 5$

$\therefore$ R.H.L. = L.H.L. = $f(2)$.

So, f is continuous at $x = 2$

Hence f has no point of discontinuity.

Q12. $f(x) = \begin{cases} x^{10} - 1, & \text{if } x \le 1 \\ x^2, & \text{if } x > 1 \end{cases}$

A.12. Given, $f(x) = \begin{cases} x^{10} - 1, & \text{if } x \le 1 \\ x^2, & \text{if } x > 1. \end{cases}$

For $x = c < 1$.

$f(c) = \lim_{x \to c} f(x) = c^{10} - 1.$

So, f is continuous for $x|>|$ 1.

For $x = c > 1$.

$f(c) = \lim_{x \to c} f(x) = c^2$

So, f is continuous for $x|>|\ 1$.

For $x = c = 1$,

L.H.L = $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^{10} - 1 - 1^{10} - 1 = 0$.

R.H.L. = $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2 = 1^2 = 1$.

$\therefore$ L.H.L $\neq$ R.H.L.

So, f is not continuous at $x = 1$.

Hence, f has point of discontinuity at $x = 1$.

Q13. Is the function defined by

$f(x) = \begin{cases} x+5, \text{ if } x \le 1 \\ x-5, \text{ if } x > 1 \end{cases}$ **a continuous function?**

Discuss the continuity of the function f, where f is defined by

A.13. Given, $f(x) = \begin{cases} x+5 \text{ if } x|\leqslant|1 \\ x-5 \text{ if } x > 1. \end{cases}$

For $x = c < 1$.

F $(c) = c + 5$

$\lim_{x\to c} f(x) = \lim_{x\to c} f\, x + 5 = c + 5$

$\therefore \lim_{x\to c} f(x) = f(c)$

So, f is continuous at $x|<|\ 1$.

For $x = c > 1$

F $(c) = c - 5$

$\lim_{x\to c} f(x) = \lim_{x\to c} x - 5 = c - 5$.

$\lim_{x \to c} f(x) = f(c)$

So, f is continuous at $x|>|\ 1$.

For $x = 1$

L.H.L. = $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x + 5 = 1 + 5 = 6$.

R.H.L. = $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} x - 5 = 1 - 5 = -4$.

L.H.L. $\neq$ R.H.L.

f is not continuous at $x = 1$

So, point of discontinuity of f is at $x = 1$.

Discussthe continuity of the function f, where f is defined by

Q14. $f(x) = \begin{cases} 3, \text{ if } \pi 0 \le x \le 1 \\ 4, \text{ if } \pi 1 < x < 3 \\ 5, \text{ if } \pi 3 \le x \le 10 \end{cases}$

A.14. Given, $f(x) = \begin{cases} 3 & \pi 0 \le x \le 1 \\ 4 & \pi 1 < x < 3 \\ 5 & \pi 3 \le x \le 10. \end{cases}$

For $x = c$ such that $0 \le c < 1$,

$f(c) = 3$

$\lim_{x \to c} f(x) = \lim_{x \to c} 3 = 3 = f(c)$

So, f is continuous in [0, 1].

For $x = c = 1$,

L.H.L. = $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 3 = 3$.

R.H.L. $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} 4 = 4$

$\therefore$ L.H.L $\neq$ R.H.L.

f is discontinuity at $x = 1$

for $x = c$ such that $1 < c < 3$.

$f(c) = 4$

$\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} 4 = 4 = f(c)$

So, f is continuous in $x \in (1,3)$

For $x = c = 3$

L.H.L. $\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^-} 4 = 4$

R.H.L. $\lim\limits_{x \to 3^+} f(x) = \lim\limits_{x \to 3^+} 5 = 5$.

So, f is discontinuous at $x = 3$.

For x = c such that $3 < c \le 10$

f (c) = 5.

$\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} 5 = 5 = f(c)$

So, f is continuous in $x \in (3,10]$

Q15. $f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \le x \le 1 \\ 4x, & \text{if } x > 1 \end{cases}$

A.15. Given $f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if} 0 \le x \le 1 \\ 4x, & \text{if } x > 1. \end{cases}$

For $(c) = c < 0$,

$f(c) = 2c.$

$\lim_{x \to c} f(x) = \lim_{x \to c} 2x = 2c = f(c)$

So, f is continuous at $x|<|\ 0$

For $x = c > 1,$

$f(c) = 4c$

$\lim_{x \to c} f(x) = \lim_{x \to c} 4x = 4c = f(c)$

So, f is continuous at $x > 1.$

For $x = 0$

L.H.L. = $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} .\ 2x = 2\ (0) = 0$

R.H.L. = $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} .\ 0 = 0.$

$f(0) = 0.$

$\therefore$ L.H.L. = R.H.L. = $f(0)$.

So, f is continuous at $x = 0$.

For $x = 1$.

L.H.L. = $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} .\ 0 = 0$

R.H.L. = $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} .\ 4x = 4\ (1) = 4.$

$\therefore$ L.H.L. $\neq$ R.H.L.

So, f is discontinuous at $x = 1$.

Q16. $f(x) = \begin{cases} -2, & \text{if } x \le -1 \\ 2x, & \text{if } -1 < x \le 1 \\ 2, & \text{if } x > 1 \end{cases}$

A.16. Given, $f(x) = \begin{cases} -2, & \text{if } x \le -1 \\ 2x, & \text{if } -1 < x \le 1 \\ 2, & \text{if } x > 1. \end{cases}$

For $x = c < -1$,

$f(c) = -2$

$\lim_{x \to c} f(x) = \lim_{x \to c} (-2) = -2 = f(c)$

So, f is continuous at $x < -1$.

For $x = c > 1$,

$f(c) = 2$

$\lim_{x \to c} f(x) = \lim_{x \to c} . 2 = 2 = f(c)$

So, f is continuous at $x |>| 1$.

For $x = -1$,

L.H.L. = $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} -2 = -2$

R.H.L. = $\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} . 2x = 2(-1) = -2$

and $f(-1) = -2$

So, L.H.L. = R.H.L. = $f(-1)$

$\therefore f$ is continuous at $x = -1$.

For $x = 1$,

L.H.L. = $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} . 2x = 2.1 = 2$

R.H.L. = $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} . 2 = 2.$

$f(1) = 2$

$f(1)$ = L.H.L = R.H.L.

So, f is continuous at $x = 1$.

Q17. Find the relationship between a and b so that the function f defined by

$$f(x) = \begin{cases} ax+1, & \text{if } x \le 3 \\ bx+3, & \text{if } x > 3 \end{cases}$$ **is continuous at $x = 3$.**

A.17. Given, $f(x) = \begin{cases} ax+1, & \text{if } x \le 3 \\ bx+3, & \text{if } x > 3 \end{cases}$ is continuous at $x = 3$

So, $f(3) = 3a + 1$

L.H.L = $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} ax + 1 = 3a + 1$

R.H.L = $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} b\,x + 3 = 3b + 3$

for continuity at $x = 3$,

L.H.L = R.H.L. = $f(3)$

$\Rightarrow 3a + 1 = 3 + 3 = 3a + 1$

So, $3a + 1 = 3b + 3$

$3a = 3b + 3 - 1$

$3a = 3b + 2.$

$a = b + \frac{2}{3}.$

Q18. Find the relationship between a and b so that the function f defined by

$$f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \le 0 \\ 4x+1, & \text{if } x > 0 \end{cases}$$ **continuous at $x = 0$? What about continuity at $x = 1$?**

A.18. Given, $f(x) = \begin{cases} \lambda(x^2 - 2x) & \text{if } x|\leqslant|0 \\ 4x+1 & \text{if } x > 0. \end{cases}$

For continuity at $x = 0$,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0).$$

$$\lim_{x\to 0^-} \lambda\left(x^2 - 2x\right) = \lim_{x\to 0^+} 4x + 1 = \lambda\left(0^2 - 2.0\right)$$

0 = 1 = 0 which is not true

Hence, f is not continuous for any value of λ.

For $x = 1$,

$$\lim_{x\to 1} f(x) = f(1).$$

$$\lim_{x\to 1} 4x + 1 = 4\,(1) + 1$$

$\Rightarrow$ 4 + 1 = 4 + 1

$\Rightarrow$ 5 = 5.

So, f is continuous at $x = 1$ $\forall$ value of λ

Q19. **Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral points. Here $[x]$ denotes the greatest integer less than or equal to x.**

A.19. Given, $g(x) = x - [x]$.

For $n \in z$,

$g(n) = n - [n] = n - n = 0$

$$\lim_{x\to n^-} f(x) = \lim_{x\to n^-} (x - [x]) = n - [n - 1] = n - n + 1 = 1$$

$$\lim_{x\to n^+} g\,(x) = \lim_{x\to n^+} x - [x] = n - [n] = 0$$

So, $\lim_{x\to n^-} g(x) \neq \lim_{x\to n^+} g(x)$.

$g(x)$ is d is continuous at all $x \in z$.

Q20. Is the function defined by $f(x) = x^2 - \sin x + 5$ continuous at $x = \pi$?

A.20. Given $f(x) = x^2 - \sin x + 5$.

At $x = \pi$.

$f(\pi) = \pi^2 - \sin\pi + 5 = \pi^2 - 0 + 5 = \pi^2 + 5.$

$\lim_{x\to\pi} f(x) = \lim_{x\to\pi} [x^2 - \sin x + 5]$

If $x = \pi + h$ then as $x \to \pi$, $h \to 0$, so,

$\lim_{x\to\pi} f(x) = \lim_{x\to 0} [(\pi + h)^2 - \sin(\pi + h) + 5]$

$= (\pi + 0)^2 - \lim_{h\to 0} [\sin\pi\cos h + \cos\pi \cdot \sin a] + 5.$

$= \pi^2 - \lim_{h\to 0} \sin\pi\cos h - \lim_{h\to 0} \cos\pi\sin h + 5$

$= x^2 - 0 \times (1) - (-1) \times 0 + 5.$

$= \pi^2 + 5 = f(x)$

So, f is continuous at $x = \pi$.

Q21. Discuss the continuity of the following functions:

(a) $f(x) = \sin x + \cos x$ (b) $f(x) = \sin x - \cos x$

(c) $f(x) = \sin x.\cos x$

A.21. (*a*) Given $f(x) = \sin x + \cos x$

(*b*). Given, $f(x) = \sin x - \cos x$

(*c*). Given, $f(x) = \sin x .\cos x$.

Let $g(x) = \sin x$ and $h(x) = \cos x$.

If g or h are continuous $f\,x$ then

g + h

g – h

g h are also continuous.

As $g(x) = \sin x$ is defined for all real number x.

Let $c \in \mathbb{R}$, and putting $x = c + h$. we see that as $x \to c, h \to 0$.

Then $g(c) = \sin c$

$\lim_{x \to c} g(x) = \lim_{x \to c} \sin x = \lim_{h \to 0} \sin (c + h)$.

$= \lim_{h \to 0} (\sin c \cos h + \cos c \sin h)$

$= \sin c . \cos 0 + \cos c . \sin 0$

$= \sin c \times 1 + 0$

$= \sin c$

$= g(c)$

So, g is continuous θ x ε R.

And $h(c) = \cos c$

$= \lim_{h \to 0} g(x) = \lim_{x \to c} \sin x = \lim_{x \to c} \cos (c + h)$

$= \cos c .\cos 0 - \sin c . \sin 0$

$= \cos c .1 - 0.$

$= \cos c = h(c)$.

As g and h are continuous θ x ε R.

The f x $f(x)$

Q22. Discuss the continuity of the cosine, cosecant, secant and cotangent functions.

A.22. For two continuous fx^n $f(x)$ and $g(x)$, $\frac{f(x)}{g(x)}, \frac{g(x)}{f(x)}$,

$\frac{1}{f(x)}$ and $\frac{1}{g(x)}$ are also continuous

Let $f(x) = \sin x$ is defined θ x ε R.

Let C E R such that $x = c + h$. so, as $x \to c$, $h \to 0$

now, $f(c) = \sin c$.

$\lim_{x \to c} f(\text{i}) = \lim_{x \to c} \sin x = \lim_{h \to 0} \sin(c + h)$.

$= \lim_{h \to 0} (\sin c \cos h + \cos c \sin h)$

$= \sin c \cos 0 + \cos c \sin 0$

$= \sin c \times 1 + 0$

$= \sin c$

$= f(c)$

So, f is continuous.

Then, $\frac{1}{f(x)}$ is also continuous

$\Rightarrow \frac{1}{\sin(x)}$ is also continuous

$\Rightarrow$ cosec x is also continuous

Let $g(x) = \cos x$ is defined θ x ε R.

Then, $g(c) = \cos c$

$\lim_{x \to c} g(x) = \lim_{x \to c} . \cos x$

$= \lim_{h \to 0} \cos(c + h)$.

$= \lim_{h \to 0} (\cos c \cos h - \sin c \sin h.)$

$= \cos c \cos h - \sin c \sin h$

$= \cos c$.

$= g(c)$

So, g is continuous

Then, $\dfrac{1}{g(x)}$ is also continuous

$\Rightarrow \dfrac{1}{\cos x}$ is also continuous

$\Rightarrow \cos x$ is also continuous.

Hence, $\dfrac{g(x)}{f(x)}$ is also continuous

$\Rightarrow \dfrac{\cos x}{\sin x}$ is also continuous

$\Rightarrow \cot x$ is also continuous .

Q23. Find all points of discontinuity of f , where

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x < 0 \\ x+1, & \text{if } x \geq 0 \end{cases}$$

A.23. Given $f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x < 0. \\ x+1, & \text{if } x \geq 0. \end{cases}$

For $x = c < 0$,

$f(c) = \dfrac{\sin c}{c}$

$$\lim_{x \to c} f(x) = \lim_{x \to c} \frac{\sin x}{x} = \frac{\sin c}{c} = f(c)$$

So, f is continuous for $x < 0$

For $x = c > 0$

$f(c) = c + 1$

$\lim_{x \to c} f(x) = \lim_{x \to c} x + 1 = c + 1 = f(c)$

So, f is continuous for $x > 0$.

For $x = 0$.

L.H.L. = $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{\sin x}{x} = 1.$

R.H.S. = $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x + 1 = 0 + 1 = 1.$

And $f(0) = 0 + 1 = 1$

$\because$ L.H.L = R.H.L. = $f(0)$

So, f is continuous at $x = 1$.

Hence, discontinuous point of x does not exit.

Q24. Determine if f defined by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$ **is a continuous function?**

A.24. Given $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0. \\ 0 \text{ if } x = 0. \end{cases}$

For $x = c \neq 0$,

$f(c) = c^2 \sin \frac{1}{c}$

$$\lim_{x \to c} f(x) = \lim_{x \to c} x^2 \sin \frac{1}{x} = c^2 \sin \frac{1}{c}.$$

So, f is continuous for $x \neq 0$.

For $x = 0$,

$f(0) = 0$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right)$$

As we have $\sin\frac{1}{x} \in [-1,1]$

$\lim_{x\to 0} f(x) = 0^2 \times a$ where $a \in [-1,1]$

$= 0 = f(0)$.

$\therefore f$ is also continuous at $x = 0$.

Q25. Examine the continuity of f, where f is defined by

$$f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \neq 0 \\ -1, & \text{if } x = 0 \end{cases}$$

A.25. Given, $f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \neq 0 \\ -1, & \text{if } x = 0. \end{cases}$

For $x = c \neq 0$,

$f(c) = \sin c - \cos c$.

$\lim_{x\to c} f(x) = \lim_{x\to c} (\sin x - \cos x) = \sin c - \cos c = f(c)$

So, f is continuous at $x \neq 0$

For $x = 0$,

$f(0) = -1$

$\lim_{x\to 0} f(x) = \lim_{x\to 0} (\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(\sin x - \text{cor}\, x) = \sin 0 - \cos 0 = -1.$$

$\therefore \lim_{x\to 0-} f(x) = \lim_{x\to 0+} f(x) = f(0)$

So, f is continuous at $x = 0$.

Find the values of k so that the function f is continuous at the indicated point in Exercises 26 to 29.

Q26. $f(x) = \begin{cases} \dfrac{k\cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases}$ at $x=\dfrac{\pi}{2}$

A.26. Given, $f(x) = \begin{cases} \dfrac{\pi\cos x}{\pi - 2x} & \text{if } x \neq \pi/2 \\ 3 & \text{if } x = \pi/2 \end{cases}$

For continuity at $x = \pi/2$

$\lim_{x\to \pi^-/2} f(x) = \lim_{x\to \pi/2} f(x) = f\left(\pi/2\right).$

$\lim_{x\to \pi/2} \dfrac{x\cos x}{\pi - 2x} = \lim_{x\to \pi/2^+} \dfrac{x\cos x}{\pi - 2x} = 3.$

Take $\lim_{x\to \pi/2} \dfrac{x\cos x}{x - 2x} = 3.$

Putting $x = \dfrac{\pi}{2} + h$ such that as $x \to \pi/2, h \to 0.$

So $\lim_{h\to 0} \dfrac{x\cos(\pi/2 + h)}{x - 2(\pi/2 + n)} = \lim_{h\to 0} \dfrac{x(-\sin x)}{-2h}$

$= \lim_{h\to 0} \dfrac{x\sin h}{2h}$

$= \dfrac{k}{2} \lim_{h\to 0} \dfrac{\sin h}{h} = \dfrac{k}{2}.$

i e, $\frac{k}{2} = 3$

$\Rightarrow$ k = 6

Similarly from $\lim\limits_{x \to \pi^+/2} \frac{x \cos x}{\pi - 2x} = 3$

$$\Rightarrow \lim_{h \to 0} \frac{h \cos\left(\frac{\pi}{2} + h\right)}{\pi - 2\left(\frac{\pi}{2} + h\right)} = \lim_{h \to 0} \frac{-h \sin h}{-2h}$$

$$= \frac{k}{2} \lim_{h \to 0} \frac{\sin h}{h}$$

$$= \frac{x}{2}$$

So, $\frac{x}{2} = 3$

$\Rightarrow$ k = 6

Q27. $f(x) = \begin{cases} kx^2, & \text{if } x \le 2 \\ 3, & \text{if } x > 2 \end{cases}$ at x=2

A.27. Given $f(x) = \begin{cases} kx^2 & \text{if } x \le 2. \\ 3 & \text{if } x > 2. \end{cases}$

For continuous at $x = 2$,

$f(2) = \text{k}\,(2)^2 = 4x.$

L.H.L. = $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} \text{k}x^2 = 4x$

R.H.L. = $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} 3 = 3$

Then, L.H.L = R.H.L. = $f(2)$

i e, $4x = 3$

$$\Rightarrow x = \frac{3}{4}.$$

Q28. $f(x) = \begin{cases} kx+1 & \text{if } x \le \pi \\ \cos x, & \text{if } x > \pi \end{cases}$ at x=π

A.28. Given, $f(x) = \begin{cases} xx+1 & \text{if } x \le \pi. \\ \cos x & \text{if } x > \pi \end{cases}$

For continuity at $x = \pi$. $f(x) = k\pi + 1$

L.H.S. = $\lim\limits_{x\to\pi^-} f(x) = \lim\limits_{x\to\pi^-} kx+1 = k\pi+1$

R.H.L. = $\lim\limits_{x\to x^+} f(x) = \lim\limits_{x\to\pi^+} \cos x = \cos x = -1.$

Then L.H.L = R.H.L. = $f(\pi)$

$\Rightarrow k\pi + 1 = -1$

$\Rightarrow k\pi = -2$

$$\Rightarrow x = -\frac{2}{\pi}.$$

Q29. $f(x) = \begin{cases} kx+1, & \text{if } x \le 5 \\ 3x-5, & \text{if } x > 5 \end{cases}$ at $x = 5$

A.29. Given, $f(x)\begin{cases} kx+1, & \text{if } x \le 5 \\ 3x-5, & \text{if } x > 5. \end{cases}$

For continuity at $x = 5$,

$\lim\limits_{x\to5^-} f(x) = \lim\limits_{x\to5^-}, kx+1 = 5x+1.$

$\lim\limits_{x\to5^+} f(x) = \lim\limits_{x\to5^+} 3x-5 = 15-5 = 10$

$f(5) = 5k + 1$

So, $\lim\limits_{x\to 5^-} f(x) = \lim\limits_{x\to 5^+} f(x) = f(5)$.

i e, 5k + 1 = 10

$\Rightarrow$ 5k = 10 –1

$\Rightarrow$ k = $\frac{9}{5}$.

Q30. Find the values of a and b such that the function defined by

$$f(x) = \begin{cases} 5, & \text{if } x \le 2 \\ ax+b, & \text{if } 2 < x < 10 \\ 21, & \text{if } x \ge 10 \end{cases}$$ **is a continuous function.**

A.30. Given, $f(x) = \begin{cases} 5 & \text{if } x \le 2 \\ ax+b & \text{if } 2 < x < 10 \\ 21 & \text{if } x \ge 10 \end{cases}$

For continuity at $x = 2$,

$\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x) = f(2)$

$\Rightarrow \lim\limits_{x\to 2^-} 5 = \lim\limits_{x\to 2^+} ax + b = 5.$

$\Rightarrow$ 5 = 2a + b — (i)

For continuous at x = 10,

$\lim\limits_{x\to 10^-} f(x) = \lim\limits_{x\to 10^+} f(x) = f(10)$

$\Rightarrow \lim\limits_{x\to 10^-} ax + b = \lim\limits_{x\to 10^+} 21 = 21.$

$\Rightarrow$ 10a + b = 21 — (2).

So, $e\ q$ (2) – 5 × $e\ q$ (1) we get,

10a + b – 5 (2a + b) = 21 – 5 × 5.

$\Rightarrow$ 10a + b – 10a– 5b = 21 – 25.

$\Rightarrow$– 4b = – 4

$\Rightarrow b = 1.$

And putting b = 1 in e q (1),

$\Rightarrow 2a = 5 - b = 5 - 1 = 4$

$\Rightarrow a = \frac{4}{2} = 2.$

Hence, $a = 2$ and b = 1.

Q31. Show that the function defined by $f(x) = \cos(x^2)$ is a continuous function

A.31. Given $f(x) = \cos(x^2)$

Let g $(x) = \cos x$ is a lregononuie fa (cosine) which is continuous function

and let $h(x) = x^2$ is a polynomial $f\,x^n$ which is also continuous

Hence (goh) x = g($h(x)$)

= g$(x)^2$

= cos (x^2)

= $f(x)$

is also a continuous $f\,x$ being a composite fx^n of how continuous $f\,x\,\forall x \in \mathbb{R}$.

Q32. Show that the function defined by $f(x) = |\cos x|$ is a continuous function.

A.32. Given, $f(x) = |\cos x|$.

Let g(x) = cos x and $h(x) = |x|$.

Hence, as cosine function and modulus $f\,x$ are continuous $\forall x \in \mathbb{R}, g$ h are continuous.

Then, (hog) $x = h$ (g(x))

= h (cos x)

$= |\cos x|$.

$= f(x)$ is also continuous being

A composites fx^n of two continuous $f\,x\;\forall x \in \mathbb{R}$.

Q33. Examine that sin $|x|$ is a continuous function.

A.33. Given, $f(x) = \sin |x|$.

Let $g(x) = \sin x$ and $h(x) = |x|$. then as sine $f\,x$ and modulus $f\,x$ are continuous in x e R

g and h are continuous.

So, $(goh)(x) = g(h(x)) = g(|x|) = \sin |x| = f(x)$

Is *a* continuous $f\,x$ being *a* competitive $f\,x$ of two continuous $f\,x$.

Q34. Find all the points of discontinuity of f defined by $f(x)=|x|-|x+1|$.

A.34. Given, $f(x)=|x|-|x+1|$

Let $g(x) = |x|$ is continuous being *a* modules $f\,x$ and $h(x) = |x+1|$ is also continuous being *a* modules $\forall x \in \mathbb{R}$

Then, $f(x) = g(x) - h(x)$.is also continuous for all x. E. R.

Hence, there is no point of discontinuous for $f(x)$.