Exercise 4.4

Question 1:

Find adjoint of each of the matrices.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Answer

$$\text{Let } A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

We have,

$$A_{11} = 4,\ A_{12} = -3,\ A_{21} = -2,\ A_{22} = 1$$

$$\therefore adjA = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

Question 2:

Find adjoint of each of the matrices.

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$$

Answer

$$\text{Let } A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}.$$

We have,

$$A_{11} = \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3 - 0 = 3$$

$$A_{12} = -\begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix} = -(2+10) = -12$$

$$A_{13} = \begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} = 0 + 6 = 6$$

$$A_{21} = -\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -(-1-0) = 1$$

$$A_{22} = \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 1+4 = 5$$

$$A_{23} = -\begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = -(0-2) = 2$$

$$A_{31} = \begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix} = -5-6 = -11$$

$$A_{32} = -\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = -(5-4) = -1$$

$$A_{33} = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 3+2 = 5$$

$$\text{Hence, } adjA = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}.$$

Question 3:

Verify A (adj A) = (adj A) A = $|A|$ I .

$$\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$$

Answer

$$A = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$$

we have,

$$|A| = -12 - (-12) = -12 + 12 = 0$$

$$\therefore |A| I = 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Now,

$$A_{11} = -6, A_{12} = 4, A_{21} = -3, A_{22} = 2$$

$$\therefore adjA = \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$$

Now,

$$A(adjA) = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix} \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -12+12 & -6+6 \\ 24-24 & 12-12 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Also, $(adjA)A = \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$

$$= \begin{bmatrix} -12+12 & -18+18 \\ 8-8 & 12-12 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, $A(adjA) = (adjA)A = |A| I$.

Question 4:

Verify A (adj A) = (adj A) A = $|A| I$.

$$\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$$

Answer

$$A=\begin{bmatrix}1 & -1 & 2\\ 3 & 0 & -2\\ 1 & 0 & 3\end{bmatrix}$$

$$|A|=1(0-0)+1(9+2)+2(0-0)=11$$

$$\therefore |A|I=11\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}11 & 0 & 0\\ 0 & 11 & 0\\ 0 & 0 & 11\end{bmatrix}$$

Now,

$A_{11}=0, A_{12}=-(9+2)=-11, A_{13}=0$

$A_{21}=-(-3-0)=3, A_{22}=3-2=1, A_{23}=-(0+1)=-1$

$A_{31}=2-0=2, A_{32}=-(-2-6)=8, A_{33}=0+3=3$

$$\therefore adjA = \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}$$

Now,

$$A(adjA) = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+11+0 & 3-1-2 & 2-8+6 \\ 0+0+0 & 9+0+2 & 6+0-6 \\ 0+0+0 & 3+0-3 & 2+0+9 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

Also,

$$(adjA)\cdot A = \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+9+2 & 0+0+0 & 0-6+6 \\ -11+3+8 & 11+0+0 & -22-2+24 \\ 0-3+3 & 0+0+0 & 0+2+9 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

Hence, $A(adjA) = (adjA)A = |A|I$.

Question 6:

Find the inverse of each of the matrices (if it exists).

$$\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$$

Answer

Let $A = \begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$.

we have,

$|A| = -2 + 15 = 13$

Now,

$A_{11} = 2, A_{12} = 3, A_{21} = -5, A_{22} = -1$

$\therefore adjA = \begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$

$\therefore A^{-1} = \dfrac{1}{|A|} adjA = \dfrac{1}{13}\begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$

Question 7:

Find the inverse of each of the matrices (if it exists).

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

Answer

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$.

We have,

$|A| = 1(10-0) - 2(0-0) + 3(0-0) = 10$

Now,

$A_{11} = 10 - 0 = 10, A_{12} = -(0-0) = 0, A_{13} = 0 - 0 = 0$

$A_{21} = -(10-0) = -10, A_{22} = 5 - 0 = 5, A_{23} = -(0-0) = 0$

$A_{31} = 8 - 6 = 2, A_{32} = -(4-0) = -4, A_{33} = 2 - 0 = 2$

$$\therefore adjA=\begin{bmatrix}10 & -10 & 2\\ 0 & 5 & -4\\ 0 & 0 & 2\end{bmatrix}$$

$$\therefore\ A^{-1}=\frac{1}{|A|}adjA=\frac{1}{10}\begin{bmatrix}10 & -10 & 2\\ 0 & 5 & -4\\ 0 & 0 & 2\end{bmatrix}$$

Question 8:

Find the inverse of each of the matrices (if it exists).

$$\begin{bmatrix}1 & 0 & 0\\ 3 & 3 & 0\\ 5 & 2 & -1\end{bmatrix}$$

Answer

Let $A=\begin{bmatrix}1 & 0 & 0\\ 3 & 3 & 0\\ 5 & 2 & -1\end{bmatrix}$.

We have,

$|A|=1(-3-0)-0+0=-3$

Now,

$A_{11}=-3-0=-3, A_{12}=-(-3-0)=3, A_{13}=6-15=-9$

$A_{21}=-(0-0)=0, A_{22}=-1-0=-1, A_{23}=-(2-0)=-2$

$A_{31}=0-0=0, A_{32}=-(0-0)=0, A_{33}=3-0=3$

$$\therefore adjA=\begin{bmatrix}-3 & 0 & 0\\ 3 & -1 & 0\\ -9 & -2 & 3\end{bmatrix}$$

$$\therefore\ A^{-1}=\frac{1}{|A|}adjA=-\frac{1}{3}\begin{bmatrix}-3 & 0 & 0\\ 3 & -1 & 0\\ -9 & -2 & 3\end{bmatrix}$$

Question 9:

Find the inverse of each of the matrices (if it exists).

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$$

Answer

Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$.

We have,

$$\begin{aligned} |A| &= 2(-1-0) - 1(4-0) + 3(8-7) \\ &= 2(-1) - 1(4) + 3(1) \\ &= -2 - 4 + 3 \\ &= -3 \end{aligned}$$

Now,

$A_{11} = -1 - 0 = -1, A_{12} = -(4-0) = -4, A_{13} = 8 - 7 = 1$

$A_{21} = -(1-6) = 5, A_{22} = 2 + 21 = 23, A_{23} = -(4+7) = -11$

$A_{31} = 0 + 3 = 3, A_{32} = -(0-12) = 12, A_{33} = -2 - 4 = -6$

$$\therefore adjA = \begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} adjA = -\frac{1}{3}\begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$$

Question 10:

Find the inverse of each of the matrices (if it exists).

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}.$$

Answer

Let $A=\begin{bmatrix}1 & -1 & 2\\ 0 & 2 & -3\\ 3 & -2 & 4\end{bmatrix}$.

By expanding along C_1, we have:

$|A|=1(8-6)-0+3(3-4)=2-3=-1$

Now,

$A_{11}=8-6=2, A_{12}=-(0+9)=-9, A_{13}=0-6=-6$

$A_{21}=-(-4+4)=0, A_{22}=4-6=-2, A_{23}=-(-2+3)=-1$

$A_{31}=3-4=-1, A_{32}=-(-3-0)=3, A_{33}=2-0=2$

$\therefore adjA=\begin{bmatrix}2 & 0 & -1\\ -9 & -2 & 3\\ -6 & -1 & 2\end{bmatrix}$

$\therefore\ A^{-1}=\frac{1}{|A|}adjA=-\begin{bmatrix}2 & 0 & -1\\ -9 & -2 & 3\\ -6 & -1 & 2\end{bmatrix}=\begin{bmatrix}-2 & 0 & 1\\ 9 & 2 & -3\\ 6 & 1 & -2\end{bmatrix}$

Question 11:

Find the inverse of each of the matrices (if it exists).

$\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\alpha & \sin\alpha\\ 0 & \sin\alpha & -\cos\alpha\end{bmatrix}$

Answer

Let $A=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\alpha & \sin\alpha\\ 0 & \sin\alpha & -\cos\alpha\end{bmatrix}$.

We have,

$|A|=1\left(-\cos^2\alpha-\sin^2\alpha\right)=-\left(\cos^2\alpha+\sin^2\alpha\right)=-1$

Now,

$A_{11}=-\cos^2\alpha-\sin^2\alpha=-1, A_{12}=0, A_{13}=0$

$A_{21}=0, A_{22}=-\cos\alpha, A_{23}=-\sin\alpha$

$A_{31}=0, A_{32}=-\sin\alpha, A_{33}=\cos\alpha$

$$\therefore adjA=\begin{bmatrix}-1 & 0 & 0\\ 0 & -\cos\alpha & -\sin\alpha\\ 0 & -\sin\alpha & \cos\alpha\end{bmatrix}$$

$$\therefore\ A^{-1}=\frac{1}{|A|}\cdot adjA=-\begin{bmatrix}-1 & 0 & 0\\ 0 & -\cos\alpha & -\sin\alpha\\ 0 & -\sin\alpha & \cos\alpha\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\alpha & \sin\alpha\\ 0 & \sin\alpha & -\cos\alpha\end{bmatrix}$$

Question 12:

Let $A=\begin{bmatrix}3 & 7\\ 2 & 5\end{bmatrix}$ and $B=\begin{bmatrix}6 & 8\\ 7 & 9\end{bmatrix}$. Verify that $(AB)^{-1}=B^{-1}A^{-1}$

Answer

Let $A=\begin{bmatrix}3 & 7\\ 2 & 5\end{bmatrix}$.

We have,

$|A|=15-14=1$

Now,

$A_{11}=5, A_{12}=-2, A_{21}=-7, A_{22}=3$

$$\therefore adjA=\begin{bmatrix}5 & -7\\ -2 & 3\end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{|A|}\cdot adjA=\begin{bmatrix}5 & -7\\ -2 & 3\end{bmatrix}$$

Now, let $B=\begin{bmatrix}6 & 8\\ 7 & 9\end{bmatrix}$.

We have,

$|B|=54-56=-2$

$\therefore adjB=\begin{bmatrix}9 & -8\\ -7 & 6\end{bmatrix}$

$$\therefore B^{-1}=\frac{1}{|B|}adjB=-\frac{1}{2}\begin{bmatrix}9 & -8\\ -7 & 6\end{bmatrix}=\begin{bmatrix}-\frac{9}{2} & 4\\ \frac{7}{2} & -3\end{bmatrix}$$

Now,

$$B^{-1}A^{-1}=\begin{bmatrix}-\frac{9}{2} & 4\\ \frac{7}{2} & -3\end{bmatrix}\begin{bmatrix}5 & -7\\ -2 & 3\end{bmatrix}$$

$$=\begin{bmatrix}-\frac{45}{2}-8 & \frac{63}{2}+12\\ \frac{35}{2}+6 & -\frac{49}{2}-9\end{bmatrix}=\begin{bmatrix}-\frac{61}{2} & \frac{87}{2}\\ \frac{47}{2} & -\frac{67}{2}\end{bmatrix} \qquad \ldots(1)$$

Then,

$$AB=\begin{bmatrix}3 & 7\\ 2 & 5\end{bmatrix}\begin{bmatrix}6 & 8\\ 7 & 9\end{bmatrix}$$

$$=\begin{bmatrix}18+49 & 24+63\\ 12+35 & 16+45\end{bmatrix}$$

$$=\begin{bmatrix}67 & 87\\ 47 & 61\end{bmatrix}$$

Therefore,we have $|AB|=67\times 61-87\times 47=4087-4089=-2.$

Also,

$$adj(AB)=\begin{bmatrix}61 & -87\\ -47 & 67\end{bmatrix}$$

$$\therefore (AB)^{-1}=\frac{1}{|AB|}adj(AB)=-\frac{1}{2}\begin{bmatrix}61 & -87\\ -47 & 67\end{bmatrix}$$

$$=\begin{bmatrix}-\frac{61}{2} & \frac{87}{2}\\ \frac{47}{2} & -\frac{67}{2}\end{bmatrix} \quad ...(2)$$

From (1) and (2), we have:

$(AB)^{-1} = B^{-1}A^{-1}$

Hence, the given result is proved.

Question 13:

If $A=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}$, show that $A^2-5A+7I=O$. Hence find A^{-1}.

Answer

$$A=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}$$

$$A^2=A\cdot A=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}=\begin{bmatrix}9-1 & 3+2\\ -3-2 & -1+4\end{bmatrix}=\begin{bmatrix}8 & 5\\ -5 & 3\end{bmatrix}$$

$$\therefore A^2-5A+7I$$

$$=\begin{bmatrix}8 & 5\\ -5 & 3\end{bmatrix}-5\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}+7\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}8 & 5\\ -5 & 3\end{bmatrix}-\begin{bmatrix}15 & 5\\ -5 & 10\end{bmatrix}+\begin{bmatrix}7 & 0\\ 0 & 7\end{bmatrix}$$

$$=\begin{bmatrix}-7 & 0\\ 0 & -7\end{bmatrix}+\begin{bmatrix}7 & 0\\ 0 & 7\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Hence, $A^2-5A+7I=O$.

$$\therefore A\cdot A-5A=-7I$$

$$\Rightarrow A\cdot A\left(A^{-1}\right)-5AA^{-1}=-7IA^{-1} \qquad \left[\text{Post-multiplying by } A^{-1} \text{ as } |A|\neq 0\right]$$

$$\Rightarrow A\left(AA^{-1}\right)-5I=-7A^{-1}$$

$$\Rightarrow AI-5I=-7A^{-1}$$

$$\Rightarrow A^{-1}=-\frac{1}{7}(A-5I)$$

$$\Rightarrow A^{-1}=\frac{1}{7}(5I-A)$$

$$=\frac{1}{7}\left(\begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix}-\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}\right)=\frac{1}{7}\begin{bmatrix}2 & -1\\ 1 & 3\end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{7}\begin{bmatrix}2 & -1\\ 1 & 3\end{bmatrix}$$

Question 14:

For the matrix $A=\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}$, find the numbers *a* and *b* such that $A^2 + aA + bI = O$.

Answer

$$A=\begin{bmatrix}3 & 2\\1 & 1\end{bmatrix}$$

$$\therefore A^2=\begin{bmatrix}3 & 2\\1 & 1\end{bmatrix}\begin{bmatrix}3 & 2\\1 & 1\end{bmatrix}=\begin{bmatrix}9+2 & 6+2\\3+1 & 2+1\end{bmatrix}=\begin{bmatrix}11 & 8\\4 & 3\end{bmatrix}$$

Now,

$$A^2+aA+bI=O$$

$$\Rightarrow (AA)A^{-1}+aAA^{-1}+bIA^{-1}=O \qquad \left[\text{Post-multiplying by } A^{-1} \text{ as } |A|\neq 0\right]$$

$$\Rightarrow A\left(AA^{-1}\right)+aI+b\left(IA^{-1}\right)=O$$

$$\Rightarrow AI+aI+bA^{-1}=O$$

$$\Rightarrow A+aI=-bA^{-1}$$

$$\Rightarrow A^{-1}=-\frac{1}{b}(A+aI)$$

Now,

$$A^{-1}=\frac{1}{|A|}adjA=\frac{1}{1}\begin{bmatrix}1 & -2\\-1 & 3\end{bmatrix}=\begin{bmatrix}1 & -2\\-1 & 3\end{bmatrix}$$

We have:

$$\begin{bmatrix}1 & -2\\-1 & 3\end{bmatrix}=-\frac{1}{b}\left(\begin{bmatrix}3 & 2\\1 & 1\end{bmatrix}+\begin{bmatrix}a & 0\\0 & a\end{bmatrix}\right)=-\frac{1}{b}\begin{bmatrix}3+a & 2\\1 & 1+a\end{bmatrix}=\begin{bmatrix}\frac{-3-a}{b} & -\frac{2}{b}\\-\frac{1}{b} & \frac{-1-a}{b}\end{bmatrix}$$

Comparing the corresponding elements of the two matrices, we have:

$$-\frac{1}{b}=-1\Rightarrow b=1$$

$$\frac{-3-a}{b}=1\Rightarrow -3-a=1\Rightarrow a=-4$$

Hence, −4 and 1 are the required values of *a* and *b* respectively.

Question 15:

For the matrix $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}$ show that $A^3 - 6A^2 + 5A + 11\,I = O$. Hence, find A^{-1}.

Answer

$$A=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}$$

$$A^2=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}1+1+2 & 1+2-1 & 1-3+3\\ 1+2-6 & 1+4+3 & 1-6-9\\ 2-1+6 & 2-2-3 & 2+3+9\end{bmatrix}=\begin{bmatrix}4 & 2 & 1\\ -3 & 8 & -14\\ 7 & -3 & 14\end{bmatrix}$$

$$A^3=A^2\cdot A=\begin{bmatrix}4 & 2 & 1\\ -3 & 8 & -14\\ 7 & -3 & 14\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}4+2+2 & 4+4-1 & 4-6+3\\ -3+8-28 & -3+16+14 & -3-24-42\\ 7-3+28 & 7-6-14 & 7+9+42\end{bmatrix}$$

$$=\begin{bmatrix}8 & 7 & 1\\ -23 & 27 & -69\\ 32 & -13 & 58\end{bmatrix}$$

$\therefore A^3 - 6A^2 + 5A + 11I$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - 6\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + 5\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} + \begin{bmatrix} 5 & 5 & 5 \\ 5 & 10 & -15 \\ 10 & -5 & 15 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Thus, $A^3 - 6A^2 + 5A + 11I = O$.

Now,

$A^3 - 6A^2 + 5A + 11I = O$

$\Rightarrow (AAA)A^{-1} - 6(AA)A^{-1} + 5AA^{-1} + 11IA^{-1} = 0$ $\quad$ $\left[\text{Post-multiplying by } A^{-1} \text{ as } |A| \neq 0\right]$

$\Rightarrow AA\left(AA^{-1}\right) - 6A\left(AA^{-1}\right) + 5\left(AA^{-1}\right) = -11\left(IA^{-1}\right)$

$\Rightarrow A^2 - 6A + 5I = -11A^{-1}$

$\Rightarrow A^{-1} = -\frac{1}{11}\left(A^2 - 6A + 5I\right) \quad \ldots(1)$

Now,

$A^2-6A+5I$

$$=\begin{bmatrix}4 & 2 & 1\\ -3 & 8 & -14\\ 7 & -3 & 14\end{bmatrix}-6\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}+5\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}4 & 2 & 1\\ -3 & 8 & -14\\ 7 & -3 & 14\end{bmatrix}-\begin{bmatrix}6 & 6 & 6\\ 6 & 12 & -18\\ 12 & -6 & 18\end{bmatrix}+\begin{bmatrix}5 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 5\end{bmatrix}$$

$$=\begin{bmatrix}9 & 2 & 1\\ -3 & 13 & -14\\ 7 & -3 & 19\end{bmatrix}-\begin{bmatrix}6 & 6 & 6\\ 6 & 12 & -18\\ 12 & -6 & 18\end{bmatrix}$$

$$=\begin{bmatrix}3 & -4 & -5\\ -9 & 1 & 4\\ -5 & 3 & 1\end{bmatrix}$$

From equation (1), we have:

$$A^{-1}=-\frac{1}{11}\begin{bmatrix}3 & -4 & -5\\ -9 & 1 & 4\\ -5 & 3 & 1\end{bmatrix}=\frac{1}{11}\begin{bmatrix}-3 & 4 & 5\\ 9 & -1 & -4\\ 5 & -3 & -1\end{bmatrix}$$

Question 16:

If $A=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$ verify that $A^3-6A^2+9A-4I=O$ and hence find A^{-1}

Answer

$$A=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$$

$$A^2=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$$

$$=\begin{bmatrix}4+1+1 & -2-2-1 & 2+1+2\\ -2-2-1 & 1+4+1 & -1-2-2\\ 2+1+2 & -1-2-2 & 1+1+4\end{bmatrix}$$

$$=\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}$$

$$A^3=A^2A=\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$$

$$=\begin{bmatrix}12+5+5 & -6-10-5 & 6+5+10\\ -10-6-5 & 5+12+5 & -5-6-10\\ 10+5+6 & -5-10-6 & 5+5+12\end{bmatrix}$$

$$=\begin{bmatrix}22 & -21 & 21\\ -21 & 22 & -21\\ 21 & -21 & 22\end{bmatrix}$$

Now,

$A^3-6A^2+9A-4I$

$$=\begin{bmatrix}22 & -21 & 21\\ -21 & 22 & -21\\ 21 & -21 & 22\end{bmatrix}-6\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}+9\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}-4\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}22 & -21 & 21\\ -21 & 22 & -21\\ 21 & -21 & 22\end{bmatrix}-\begin{bmatrix}36 & -30 & 30\\ -30 & 36 & -30\\ 30 & -30 & 36\end{bmatrix}+\begin{bmatrix}18 & -9 & 9\\ -9 & 18 & -9\\ 9 & -9 & 18\end{bmatrix}-\begin{bmatrix}4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{bmatrix}$$

$$=\begin{bmatrix}40 & -30 & 30\\ -30 & 40 & -30\\ 30 & -30 & 40\end{bmatrix}-\begin{bmatrix}40 & -30 & 30\\ -30 & 40 & -30\\ 30 & -30 & 40\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$\therefore A^3-6A^2+9A-4I=O$

Now,

$A^3-6A^2+9A-4I=O$

$\Rightarrow (AAA)A^{-1}-6(AA)A^{-1}+9AA^{-1}-4IA^{-1}=O$ $\qquad$ $\left[\text{Post-multiplying by } A^{-1} \text{ as } |A|\neq 0\right]$

$\Rightarrow AA\left(AA^{-1}\right)-6A\left(AA^{-1}\right)+9\left(AA^{-1}\right)=4\left(IA^{-1}\right)$

$\Rightarrow AAI-6AI+9I=4A^{-1}$

$\Rightarrow A^2-6A+9I=4A^{-1}$

$\Rightarrow A^{-1}=\dfrac{1}{4}\left(A^2-6A+9I\right)$ $\qquad \ldots(1)$

$A^2-6A+9I$

$$=\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}-6\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}+9\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$=\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}-\begin{bmatrix}12 & -6 & 6\\ -6 & 12 & -6\\ 6 & -6 & 12\end{bmatrix}+\begin{bmatrix}9 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 9\end{bmatrix}$$

$$=\begin{bmatrix}3 & 1 & -1\\ 1 & 3 & 1\\ -1 & 1 & 3\end{bmatrix}$$

From equation (1), we have:

$$A^{-1}=\frac{1}{4}\begin{bmatrix}3 & 1 & -1\\ 1 & 3 & 1\\ -1 & 1 & 3\end{bmatrix}$$

Question 17:

Let A be a nonsingular square matrix of order 3 × 3. Then $|adjA|$ is equal to

A. $|A|$ **B.** $|A|^2$ **C.** $|A|^3$ **D.** $3|A|$

Answer **B**

We know that,

$$(adjA)A=|A|I=\begin{bmatrix}|A| & 0 & 0\\ 0 & |A| & 0\\ 0 & 0 & |A|\end{bmatrix}$$

$$\Rightarrow |(adjA)A|=\begin{vmatrix}|A| & 0 & 0\\ 0 & |A| & 0\\ 0 & 0 & |A|\end{vmatrix}$$

$$\Rightarrow |adjA||A|=|A|^3\begin{vmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{vmatrix}=|A|^3(I)$$

$$\therefore |adjA|=|A|^2$$

Hence, the correct answer is B.

Question 18:

If A is an invertible matrix of order 2, then det (A^{-1}) is equal to

A. det (A) **B.** $\frac{1}{\det(A)}$ **C.** 1 **D.** 0

Answer

Since A is an invertible matrix, A^{-1} exists and $A^{-1}=\frac{1}{|A|}adjA.$

As matrix A is of order 2, let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Then, $|A| = ad - bc$ and $adjA = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Now,

$$A^{-1} = \frac{1}{|A|} adjA = \begin{bmatrix} \frac{d}{|A|} & \frac{-b}{|A|} \\ \frac{-c}{|A|} & \frac{a}{|A|} \end{bmatrix}$$

$$\therefore \left|A^{-1}\right| = \begin{vmatrix} \frac{d}{|A|} & \frac{-b}{|A|} \\ \frac{-c}{|A|} & \frac{a}{|A|} \end{vmatrix} = \frac{1}{|A|^2}\begin{vmatrix} d & -b \\ -c & a \end{vmatrix} = \frac{1}{|A|^2}(ad-bc) = \frac{1}{|A|^2}.|A| = \frac{1}{|A|}$$

$$\therefore \det\left(A^{-1}\right) = \frac{1}{\det(A)}$$

Hence, the correct answer is B.