## Exercise 4.3

**Question 1:**

Write Minors and Cofactors of the elements of following determinants:

(i) $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$ (ii) $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

Answer

(i) The given determinant is $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$.

Minor of element $a_{ij}$ is $M_{ij}$.

$\therefore M_{11}$ = minor of element $a_{11} = 3$

$M_{12}$ = minor of element $a_{12} = 0$

$M_{21}$ = minor of element $a_{21} = -4$

$M_{22}$ = minor of element $a_{22} = 2$

Cofactor of $a_{ij}$ is $A_{ij} = (-1)^{i+j} M_{ij}$.

$\therefore A_{11} = (-1)^{1+1} M_{11} = (-1)^2 (3) = 3$

$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 (0) = 0$

$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 (-4) = 4$

$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 (2) = 2$

(ii) The given determinant is $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$.

Minor of element $a_{ij}$ is $M_{ij}$.

$\therefore M_{11}$ = minor of element $a_{11} = d$

$M_{12}$ = minor of element $a_{12} = b$

$M_{21}$ = minor of element $a_{21} = c$

$M_{22}$ = minor of element $a_{22} = a$

Cofactor of $a_{ij}$ is $A_{ij} = (-1)^{i+j} M_{ij}$.

$\therefore A_{11} = (-1)^{1+1} M_{11} = (-1)^2 (d) = d$

$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 (b) = -b$

$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 (c) = -c$

$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 (a) = a$

**Question 2:**

(i) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

Answer

(i) The given determinant is $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$.

By the definition of minors and cofactors, we have:

$M_{11}$ = minor of $a_{11}$= $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

$M_{12}$ = minor of $a_{12}$= $\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$

$M_{13}$ = minor of $a_{13}$ = $\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$

$M_{21}$ = minor of $a_{21}$ = $\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$

$M_{22}$ = minor of $a_{22}$ = $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

$M_{23}$ = minor of $a_{23}$ = $\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$

$M_{31}$ = minor of $a_{31}$= $\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0$

$M_{32}$ = minor of $a_{32}$ = $\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$

$M_{33}$ = minor of $a_{33}$ = $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

$A_{11}$ = cofactor of $a_{11}$= $(-1)^{1+1}$ $M_{11}$ = 1

$A_{12}$ = cofactor of $a_{12}$ = $(-1)^{1+2}$ $M_{12}$ = 0

$A_{13}$ = cofactor of $a_{13}$ = $(-1)^{1+3}$ $M_{13}$ = 0

$A_{21}$ = cofactor of $a_{21}$ = $(-1)^{2+1}$ $M_{21}$ = 0

$A_{22}$ = cofactor of $a_{22}$ = $(-1)^{2+2}$ $M_{22}$ = 1

$A_{23}$ = cofactor of $a_{23}$ = $(-1)^{2+3}$ $M_{23}$ = 0

$A_{31}$ = cofactor of $a_{31}$ = $(-1)^{3+1}$ $M_{31}$ = 0

$A_{32}$ = cofactor of $a_{32}$ = $(-1)^{3+2}$ $M_{32}$ = 0

$A_{33}$ = cofactor of $a_{33}$ = $(-1)^{3+3}$ $M_{33}$ = 1

(ii) The given determinant is $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$.

By definition of minors and cofactors, we have:

$M_{11}$ = minor of $a_{11}$= $\begin{vmatrix} 5 & -1 \\ 1 & 2 \end{vmatrix} = 10+1 = 11$

$M_{12}$ = minor of $a_{12}$= $\begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix} = 6 - 0 = 6$

$M_{13}$ = minor of $a_{13}$ = $\begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3 - 0 = 3$

$M_{21}$ = minor of $a_{21}$ = $\begin{vmatrix} 0 & 4 \\ 1 & 2 \end{vmatrix} = 0 - 4 = -4$

$M_{22}$ = minor of $a_{22}$ = $\begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2$

$M_{23}$ = minor of $a_{23}$ = $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$

$M_{31}$ = minor of $a_{31}$= $\begin{vmatrix} 0 & 4 \\ 5 & -1 \end{vmatrix} = 0 - 20 = -20$

$M_{32}$ = minor of $a_{32}$ = $\begin{vmatrix} 1 & 4 \\ 3 & -1 \end{vmatrix} = -1 - 12 = -13$

$M_{33}$ = minor of $a_{33}$ = $\begin{vmatrix} 1 & 0 \\ 3 & 5 \end{vmatrix} = 5 - 0 = 5$

$A_{11}$ = cofactor of $a_{11}$= $(-1)^{1+1}$ $M_{11}$ = 11

$A_{12}$ = cofactor of $a_{12}$ = $(-1)^{1+2}$ $M_{12}$ = −6

$A_{13}$ = cofactor of $a_{13}$ = $(-1)^{1+3}$ $M_{13}$ = 3

$A_{21}$ = cofactor of $a_{21}$ = $(-1)^{2+1}$ $M_{21}$ = 4

$A_{22}$ = cofactor of $a_{22}$ = $(-1)^{2+2}$ $M_{22}$ = 2

$A_{23}$ = cofactor of $a_{23}$ = $(-1)^{2+3}$ $M_{23}$ = −1

$A_{31}$ = cofactor of $a_{31}$ = $(-1)^{3+1}$ $M_{31}$ = −20

$A_{32}$ = cofactor of $a_{32}$ = $(-1)^{3+2}$ $M_{32}$ = 13

$A_{33}$ = cofactor of $a_{33}$ = $(-1)^{3+3}$ $M_{33}$ = 5

**Question 3:**

Using Cofactors of elements of second row, evaluate $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$.

Answer

The given determinant is $\begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$.

We have:

$M_{21} = \begin{vmatrix} 3 & 8 \\ 2 & 3 \end{vmatrix} = 9 - 16 = -7$

$\therefore A_{21}$ = cofactor of $a_{21} = (-1)^{2+1} M_{21} = 7$

$M_{22} = \begin{vmatrix} 5 & 8 \\ 1 & 3 \end{vmatrix} = 15 - 8 = 7$

$\therefore A_{22}$ = cofactor of $a_{22} = (-1)^{2+2} M_{22} = 7$

$M_{23} = \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 10 - 3 = 7$

$\therefore A_{23}$ = cofactor of $a_{23} = (-1)^{2+3} M_{23} = -7$

We know that Δ is equal to the sum of the product of the elements of the second row with their corresponding cofactors.

$\therefore \Delta = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} = 2(7) + 0(7) + 1(-7) = 14 - 7 = 7$

**Question 4:**

Using Cofactors of elements of third column, evaluate $\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$

Answer

The given determinant is $\begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$.

We have:

$M_{13} = \begin{vmatrix} 1 & y \\ 1 & z \end{vmatrix} = z - y$

$M_{23} = \begin{vmatrix} 1 & x \\ 1 & z \end{vmatrix} = z - x$

$M_{33} = \begin{vmatrix} 1 & x \\ 1 & y \end{vmatrix} = y - x$

$\therefore A_{13}$ = cofactor of $a_{13} = (-1)^{1+3} M_{13} = (z - y)$

$A_{23}$ = cofactor of $a_{23} = (-1)^{2+3} M_{23} = -(z - x) = (x - z)$

$A_{33}$ = cofactor of $a_{33} = (-1)^{3+3} M_{33} = (y - x)$

We know that Δ is equal to the sum of the product of the elements of the second row with their corresponding cofactors.

$$\therefore \Delta = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$$
$$= yz(z-y) + zx(x-z) + xy(y-x)$$
$$= yz^2 - y^2z + x^2z - xz^2 + xy^2 - x^2y$$
$$= (x^2z - y^2z) + (yz^2 - xz^2) + (xy^2 - x^2y)$$
$$= z(x^2 - y^2) + z^2(y-x) + xy(y-x)$$
$$= z(x-y)(x+y) + z^2(y-x) + xy(y-x)$$
$$= (x-y)\left[zx + zy - z^2 - xy\right]$$
$$= (x-y)\left[z(x-z) + y(z-x)\right]$$
$$= (x-y)(z-x)[-z+y]$$
$$= (x-y)(y-z)(z-x)$$

Hence, $\Delta = (x-y)(y-z)(z-x)$.

**Question 5:**

For the matrices *A* and *B*, verify that (*AB*)′ = $B'A'$ where

(i) $A = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 5 & 7 \end{bmatrix}$

Answer

**(i)** $AB = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix} \begin{bmatrix} -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 1 \\ 4 & -8 & -4 \\ -3 & 6 & 3 \end{bmatrix}$

$\therefore (AB)' = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix}$

Now, $A' = \begin{bmatrix} 1 & -4 & 3 \end{bmatrix}$, $B' = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

$\therefore B'A' = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix}$

Hence, we have verified that $(AB)' = B'A'$.

**(ii)** $AB = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 5 & 7 \\ 2 & 10 & 14 \end{bmatrix}$

$\therefore (AB)' = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 5 & 10 \\ 0 & 7 & 14 \end{bmatrix}$

Now, $A' = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}, B' = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}$

$\therefore B'A' = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 5 & 10 \\ 0 & 7 & 14 \end{bmatrix}$

Hence, we have verified that $(AB)' = B'A'$.