Exercise 10.4

Q.1. Find $\left|\vec{a} \, x \, \vec{b}\right|$ **if** $\vec{a} = \hat{i} - 7\hat{j} = 7\hat{k}$ **and** $\vec{b} = 3\hat{i} - 2\hat{j} = 2\hat{k}$

A.1. We have,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -7 & 7 \\ 3 & -2 & 2 \end{vmatrix}$$

$$\vec{a} \times \vec{b} =$$
$$= \left(-14 - (-14)\right)\hat{i} - (2 - 21)\hat{j} + \left(-2 - (-21)\right)\hat{k}$$
$$= (-14 + 14)\hat{i} - (-19)\hat{j} + (-2 + 21)\hat{k}$$
$$= 0 + 19\hat{j} + 19\hat{k}$$

Hence,

$$\left|\vec{a} \times \vec{b}\right| = \sqrt{(19)^2 + (19)^2}$$
$$= \sqrt{2 \times (19)^2} = 19\sqrt{2}$$

Q2. Find a unit vector perpendicular to each of the vectors $\vec{a} + \vec{b} \text{ and } \vec{a} - \vec{b}$, where $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} - 2\hat{k}$

A.2. Given,

$$\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$$
$$\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$$
$$\vec{a} + \vec{b} = 4\hat{i} + 4\hat{j}, \vec{a} - \vec{b} = 2\hat{i} + 4\hat{j}$$

A vector which is perpendicular to both $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ is given by

$$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix}$$

Say

$$\vec{c} = \hat{i}(16-0) - \hat{j}(16-0) + \hat{k}(0-8)$$
$$\vec{c} = 16\hat{i} - 16\hat{j} - 8\hat{k}$$
$$|\vec{c}| = \sqrt{(16)^2 + (-16)^2 + (-8)^2}$$
$$= \sqrt{8^2 \times 2^2 + (-2)^2 \times 8^2 + (-8)^2}$$
$$= \sqrt{8^2\left(2^2 + (-2)^2 + (-1)^2\right)}$$
$$= 8\sqrt{4+4+1} = 8\sqrt{9}$$
$$= 8 \times 3 = 24$$

Therefore, the unit vector is

$$\frac{\vec{c}}{|\vec{c}|} = \pm\frac{16\hat{i} - 16\hat{j} - 8\hat{k}}{24}$$
$$= \pm\frac{16}{24}\hat{i} \pm \frac{16}{24}\hat{j} \pm \frac{8}{24}\hat{k}$$
$$= \pm\frac{2}{3}\hat{i} \pm \frac{2}{3}\hat{j} \pm \frac{1}{3}\hat{k}$$

Q3. If a unit vector $\vec{a}$ makes an angle π/3 with $\hat{i}, \frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$ then find θ and hence, the components of $\vec{a}$.

A3. Let $\vec{a} = (a_1, a_2, a_3)$ as component

We know,

$\vec{a}$ is a unit vector, $|\vec{a}| = 1$

Given that,

$\vec{a}$ marks angles $\frac{\pi}{3}$ with $\hat{i}, \frac{\pi}{4}$ with $\hat{j}$ and θ with $\hat{k}$ acute angle.

Now,

$$\cos\frac{\pi}{3} = \frac{a_1}{|\vec{a}|} \Rightarrow \frac{1}{2} = a_1 \left[|\vec{a}| = 1\right]$$
$$\cos\frac{\pi}{4} = \frac{a_2}{|\vec{a}|} \Rightarrow \frac{1}{\sqrt{2}} = a_2$$
$$\cos\theta = \frac{a_3}{|\vec{a}|} \Rightarrow a_3 = \cos\theta$$

We know,

$$|\vec{a}| = 1$$

$$\Rightarrow \sqrt{(a_1)^2+(a_2)^2+(a_3)^2}=1$$

$$\Rightarrow \sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2+\cos^2\theta}=1$$

$$\Rightarrow \frac{1}{4}+\frac{1}{2}+\cos^2\theta=1$$

$$\Rightarrow \cos^2=1-\frac{1}{4}-\frac{1}{2}$$

$$=\frac{4-1-2}{4}=\frac{1}{4}$$

$$\therefore \cos\theta=\frac{1}{2}$$

$$\Rightarrow \theta=\frac{\pi}{3}$$

$$\therefore a_2=\cos\frac{\pi}{3}=\frac{1}{2}$$

Therefore, the component of $\vec{a}$ are $\left(\frac{1}{2},\frac{1}{\sqrt{2}},\frac{1}{2}\right)$

Q.4. Show that $\left(\vec{a}-\vec{b}\right)\times\left(\vec{a}+\vec{b}\right)=2\left(\vec{a}\times\vec{b}\right)$

A4. Show that

$$\left(\vec{a}-\vec{b}\right)\times\left(\vec{a}+\vec{b}\right)=2\left(\vec{a}\times\vec{b}\right)$$

$$\left(\vec{a}-\vec{b}\right)\times\left(\vec{a}+\vec{b}\right)$$

$$=\vec{a}\left(\vec{a}+\vec{b}\right)-\vec{b}\left(\vec{a}+\vec{b}\right)$$

$$=\vec{a}\times\vec{a}+\vec{a}\times\vec{b}-\vec{b}\times\vec{a}-\vec{b}\times\vec{b}$$

$$=0+\vec{a}\times\vec{b}-\vec{b}\times\vec{a}-0$$

$$=\vec{a}\times\vec{b}+\vec{a}\times\vec{b}\left[\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}\right]$$

$$=2(\vec{a}\times\vec{b})$$

Q.5. Find λ and μ if $\left(\vec{a}-\vec{b}\right)\times\left(\vec{a}+\vec{b}\right)=2\left(\vec{a}\times\vec{b}\right)$

A5. $\left(2\hat{i}+6\hat{j}+27\hat{k}\right)\times\left(\hat{i}+\lambda\hat{j}+\mu\hat{k}\right)=\vec{0}$

$$\begin{vmatrix} \hat{i} & \hat{j} & k \\ 2 & 6 & 27 \\ 1 & \lambda & \mu \end{vmatrix} = 0\hat{i}+0\hat{j}+0\hat{k}$$

$$\Rightarrow \hat{i}\left(6\mu-27\lambda\right)-\hat{j}\left(2\mu-27\right)+\hat{k}\left(2\lambda-6\right)$$

$$=0\hat{i}+0\hat{j}+0\hat{k}$$

On comparing both side components,

$$6\mu - 27\lambda = 0,$$
$$2\mu - 27 = 0$$
$$2\mu = 27$$
$$\mu = \frac{27}{2},$$
$$2\lambda - 6 = 0$$
$$2\lambda = 6$$
$$\lambda = \frac{6}{2} = 3$$

$\therefore$ Therefore, the value of $\mu = \frac{27}{2}$ and $\lambda = 3$.

Q6. Given that $\vec{a}.\vec{b} = 0$ and $\vec{a} \times \vec{b} = 0$ What can you conclude about the vectors $\vec{a}$ and $\vec{b}$?

A.6. Given,

$\vec{a}.\vec{b} = 0$ and $\vec{a} \times \vec{b} = 0$

For,

$\vec{a}.\vec{b} = 0$, then either $|\vec{a}| = 0$ or $|\vec{b}| = 0$ or $\vec{a} \perp \vec{b}$

For,

$\vec{a} \times \vec{b} = 0$, then either $|\vec{a}| = 0$ or $|\vec{b}| = 0$ or $\vec{a} \parallel \vec{b}$

$\therefore$ In case $\vec{a}$ and $\vec{b}$ are non- zero on both side.

But $\vec{a}$ and $\vec{b}$ cannot be both perpendicular and parallel simultaneously.

So, we can conclude that

$|\vec{a}| = 0$ or $|\vec{b}| = 0$

Q7. Let the vectors $\vec{a}, \vec{b}, \vec{c}$ be given as $a_1\hat{i} + a_2\hat{j} + a_3\hat{k},\ b_1\hat{i} + b_2\hat{j} + b_3\hat{k},\ c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ then show that $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

A.7. Given,

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

$\therefore \left(\vec{b} + \vec{c}\right) = \left(b_1 + c_1\right)\hat{i} + \left(b_2 + c_2\right)\hat{j} + \left(b_3 + c_3\right)\hat{k}$

Now,

$$\vec{a} \times \left(\vec{b} + \vec{c}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \end{vmatrix}$$

$= \hat{i}\left\{a_2\left(b_2 + c_3\right) - a_3\left(b_2 + c_2\right)\right\} - \hat{j}\left\{a_1\left(b_3 + c_3\right) - a_3\left(b_1 + c_1\right)\right\}$

$+\hat{k}\left\{a_1\left(b_2 + c_2\right) - a_2\left(b_1 + c_1\right)\right\}$

$= \hat{i}\left\{a_2b_2 + a_2c_3 - a_3b_2 - a_3c_2\right\} - \hat{j}\left\{a_1b_3 + a_1c_3 - a_3b_1 - a_3c_1\right\}$

$+\hat{k}\left\{a_1b_2 + a_1c_2 - a_2b_1 - a_2c_2\right\} ----(1)$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$= \hat{i}\left(a_2b_3 - a_3b_2\right) - \hat{j}\left(a_1b_3 - a_3b_1\right) + \hat{k}\left(a_1b_2 - a_2b_1\right) ----(2)$

And,

$$\vec{a} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$= \hat{i}\left(a_2c_3 - a_3c_2\right) - \hat{j}\left(a_1c_3 - a_3c_1\right) + \hat{k}\left(a_1c_2 - a_2c_1\right) ----(3)$

Adding (2) and (3), we get

$\left(\vec{a} \times \vec{b}\right) + \left(\vec{a} \times \vec{c}\right) = \hat{i}\left(a_2b_3 - a_3b_2\right) - \hat{j}\left(a_1b_3 - a_3b_1\right) + \hat{k}\left(a_1b_2 - a_2b_1\right)$

$+\hat{i}\left(a_2c_3 - a_3c_2\right) - \hat{j}\left(a_1c_3 - a_3c_1\right) + \hat{k}\left(a_1c_2 - a_2c_1\right)$

$\left(\vec{a} \times \vec{b}\right) + \left(\vec{a} \times \vec{c}\right) = \hat{i}\left(a_2b_3 - a_3b_2 + a_2c_3 - a_3c_2\right) + \hat{j}\left(-a_1b_3 + a_3b_1 - a_1c_3 + a_3c_1\right)$

$+\hat{k}\left(a_1b_2 - a_2b_1 + a_1c_2 - a_2c_1\right)$

$= \hat{i}\left(a_2b_3 + a_2c_3 - a_3c_2 - a_3b_2\right) - \hat{j}\left(a_1b_3 + a_1c_3 - a_3b_1 - a_3c_1\right)$

$+\hat{k}\left(a_1b_2 + a_1c_2 - a_2b_1 - a_2c_1\right) -----(4)$

From (1) and (4), we have

$\vec{a}\left(\vec{b} + \vec{c}\right) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

Hence, proved.

Q8. If either $\vec{a} = 0$ and $\vec{b} = 0$ then $\vec{a}\times\vec{b}=0$ Is the converse true? Justify your answer with an example.

A.8. We take any parallel non- zero vectors so that $\vec{a}\times\vec{b}=0$.

Let

$$\vec{a}=2\hat{i}+3\hat{j}+4\hat{k}$$
$$\vec{b}=4\hat{i}+6\hat{j}+8\hat{k}$$
Then,

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 4\\ 4 & 6 & 8\end{vmatrix}$$

$$=\hat{i}\left(2.4-2.4\right)-\hat{j}\left(12-12\right)+\hat{k}\left(12-12\right)=\vec{0}$$
Now,
$$|\vec{a}|=\sqrt{(2)^2+(3)^2+(4)^2}=\sqrt{4+9+16}=\sqrt{29}$$
$$\therefore \vec{a}\neq 0$$
$$|\vec{b}|=\sqrt{(4)^2+(6)^2+(8)^2}=\sqrt{16+36+64}=\sqrt{116}$$
$$\therefore \vec{b}\neq 0$$

Therefore, the converse of given statement need not to be true.

Q9. Find the area of the triangle with vertices A (1, 1, 2), B (2, 3, 5) and C (1, 5, 5).

A.9. Given,

$$A(1,1,2), B(2,3,5) C(1,5,5)$$

We have,

$$\vec{AB}=\hat{i}+2\hat{j}+3\hat{k}$$
$$\vec{AC}=4\hat{j}+3\hat{k}$$

The area of given triangle is $\frac{1}{2}\left|\vec{AB}\times\vec{AC}\right|$

$$\left|\vec{AB}\times\vec{AC}\right|=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 3\\ 0 & 4 & 3\end{vmatrix}$$

$$\left|\vec{AB}\times\vec{AC}\right| =$$
$$= \hat{i}(6-12) - \hat{j}(3-0) + \hat{k}(4-0)$$
$$= -6\hat{i} - 3\hat{j} + 4\hat{k}$$

Therefore,

$$\left|\vec{AB}\times\vec{AC}\right| = \sqrt{(-6)^2 + (-3)^2 + (4)^2}$$
$$= \sqrt{36+9+16}$$
$$= \sqrt{61}$$

Thus, the reQ.uired area = $\frac{1}{2}\times\sqrt{61}$

$$= \frac{\sqrt{61}}{2}$$

Q10. Find the area of the parallelogram whose adjacent sides are determined by the vectors $a = \vec{i} - \vec{j} + 3\vec{k}$ and $b = 2\vec{i} - 7\vec{j} + \vec{k}$

A.10. Given,

$$\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$$
$$\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$$

The area of a parallelogram with $\vec{a}$ and $\vec{b}$ as its adjacent sides is given by $\left|\vec{a}\times\vec{b}\right|$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{vmatrix}$$

$$\vec{a}\times\vec{b} =$$
$$= \hat{i}\left(-1-(-21)\right) - \hat{j}(1-6) + \hat{k}\left(-2-(-7)\right)$$
$$= \hat{i}(-1+21) - \hat{j}(-5) + \hat{k}(-2+7)$$
$$= 20\hat{i} + 5\hat{j} + 5\hat{k}$$

Therefore,

$$\left|\vec{a}\times\vec{b}\right| = \sqrt{(20)^2 + (5)^2 + (5)^2}$$
$$= \sqrt{400+25+25}$$
$$= \sqrt{450} = \sqrt{15\times15\times2}$$
$$= 15\sqrt{2}$$

Q11. Let the vectors $\vec{a}$ and $\vec{b}$ such that $|\vec{a}| = 3$ and $|\vec{b}| = \sqrt{2}/3$ then $\vec{a}\times\vec{b}$ is a unit vector, if the angle between $\vec{a}$ and $\vec{b}$ is:

(A) $\frac{\pi}{6}$

(B) $\frac{\pi}{4}$

(C) $\frac{\pi}{3}$

(D) $\frac{\pi}{2}$

A.11. (B) Given,

$$|\vec{a}| = 3, |\vec{b}| = \frac{\sqrt{2}}{3}$$

We know that $\vec{a} \times \vec{b} = |\vec{a}| \times |\vec{b}| \sin\theta \hat{n}$

Now,

$\vec{a} \times \vec{b}$ is a unit vector if $|\vec{a} \times \vec{b}| = 1, so$

$$\Rightarrow |\vec{a} \times \vec{b}| = 1$$

$$\Rightarrow ||\vec{a}||\vec{b}|| |\sin\theta \hat{n}| = 1$$

$$\Rightarrow 3 \times \frac{\sqrt{2}}{3} \times \sin\theta = 1$$

$$\Rightarrow \sin\theta = \frac{1}{\sqrt{2}} = \frac{\pi}{4}$$

$\therefore$ Hence, $\vec{a} \times \vec{b}$ is a unit vector if angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{4}$

Q12. Area of a rectangle having vertices A, B, C and D with position vectors $-\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}, \hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}, \text{and} -\hat{i}\frac{1}{2}\hat{j} + 4\hat{k}$ **respectively is:**

(A) $\frac{1}{2}$

(B) 1

(C) 2

(D) 4

A.12. (C)

Given,

$$A = -\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}$$

$$B = \hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}$$

$$C = \hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$$

$$D = -\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$$

$$A\vec{B} = (1+1)\hat{i} + \left(\frac{1}{2} - \frac{1}{2}\right)\hat{j} + (4-4)\hat{k}$$

$$= 2\hat{i}$$

$$B\vec{C} = (1-1)\hat{i} + \left(-\frac{1}{2} - \frac{1}{2}\right)\hat{j} + (4-4)\hat{k}$$

$$= -\hat{j}$$

$$\therefore \left|\vec{AB} \times \vec{BC}\right| = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 0 \\ 0 & -1 & 0 \end{vmatrix} = -2\hat{k}$$

$$\left|A\vec{B} \times B\vec{C}\right| = \sqrt{(-2)^2} = 2$$