Exercise 10.3

Q1. Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitude √3 and 2 respectively having $\vec{a} \cdot \vec{b} = \sqrt{6}$

A.1. Given,

$$|\vec{a}| = \sqrt{3}, |\vec{b}| = 2 and \vec{a}.\vec{b} = \sqrt{6}$$

We have,

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta.$$

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{\sqrt{6}}{\sqrt{3}\times 2} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$\cos\theta = \frac{1}{\sqrt{2}};\theta = \frac{\pi}{4}$$

Thus, angle between the given vectors $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{4}$.

Q2. Find the angle between the vectors $\hat{i} - 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$.

A.2. Given vectors are:

$$\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$$
$$\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$$
$$|\vec{a}| = \sqrt{(1)^2 + (-2)^2 + (3)^2} = \sqrt{1+4+9} = \sqrt{14}$$
$$|\vec{b}| = \sqrt{(3)^2 + (-2)^2 + (1)^2} = \sqrt{9+4+1} = \sqrt{14}$$

Now,

$$\vec{a}.\vec{b} = (\hat{i} - 2\hat{j} + 3\hat{k})(3\hat{i} - 2\hat{j} + \hat{k})$$
$$= 1.3 + (-2).(-2) + 3.1$$
$$= 3 + 4 + 3 = 10$$

Also, we know,

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{10}{\sqrt{14}\sqrt{14}} = \frac{10}{14} = \frac{5}{7}$$

$$\theta = \cos^{-1}\left(\frac{5}{7}\right)$$

Hence, the angle between the vectors is $\cos^{-1}\left(\frac{5}{7}\right)$

Q3. Find the projection of the vector $\hat{i} - \hat{j}$ on the vector $\hat{i} + \hat{j}$

A.3. Let,

$$\vec{a} = \hat{i} - \hat{j}$$
$$\vec{b} = \hat{i} + \hat{j}$$

The projection of vector $\vec{a}$ on $\vec{b}$ is given by,

$$\frac{1}{|\vec{b}|}(\vec{a}.\vec{b}) = \frac{1}{\sqrt{1+1}}(\hat{i}-\hat{j}).(\hat{i}+\hat{j})$$

$$= \frac{1}{\sqrt{2}}(1-1) = 0$$

$\therefore$ The projection of vector $\vec{a}$ on $\vec{b}$ is 0.

Q4. Find the projection of the vector $\hat{i}+3\hat{j}+7\hat{k}$ **on the vector** $7\hat{i}-\hat{j}+8\hat{k}$

A.4. Let,

$$\vec{a} = \hat{i}+3\hat{j}+7\hat{k}$$
$$\vec{b} = 7\hat{i}-\hat{j}+8\hat{k}$$

The project of vector $\vec{a}$ on $\vec{b}$ is.

$$\frac{1}{|\vec{b}|}(\vec{a}.\vec{b}) = \frac{1}{\sqrt{(7)^2+(-1)^2+(8)^2}}(\hat{i}+3\hat{j}+7\hat{k})(7\hat{i}-\hat{j}+8\hat{k})$$

$$= \frac{1}{\sqrt{49+1+64}}(7-3+56)$$

$$= \frac{60}{\sqrt{114}}$$

Q5. Show that each of the given three vectors is a unit vector:

$$\frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{j}), \frac{1}{7}(3\hat{i}+6\hat{j}+2\hat{k}), \frac{1}{7}(6\hat{i}+2\hat{j}+3\hat{k})$$

A.5. Let

$$\vec{a} = \frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k}) = \frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}+\frac{6}{7}\hat{k}$$

$$\vec{b} = \frac{1}{7}(3\hat{i}-6\hat{j}+2\hat{k}) = \frac{3}{7}\hat{i}+\frac{-6}{7}\hat{j}+\frac{2}{7}\hat{k}$$

$$\vec{c} = \frac{1}{7}(6\hat{i}+2\hat{j}-3\hat{k}) = \frac{6}{7}\hat{i}+\frac{2}{7}\hat{j}-\frac{3}{7}\hat{k}$$

$$|\vec{a}| = \sqrt{\left(2/7\right)^2+\left(3/7\right)^2+\left(6/7\right)^2} = \sqrt{\frac{4}{49}+\frac{9}{49}+\frac{36}{49}} = \sqrt{\frac{49}{49}} = 1$$

$$|\vec{b}| = \sqrt{\left(3/7\right)^2+\left(-6/7\right)^2+\left(2/7\right)^2} = \sqrt{\frac{9}{49}+\frac{36}{49}+\frac{4}{49}} = \sqrt{\frac{49}{49}} = 1$$

$$|\vec{c}| = \sqrt{\left(6/7\right)^2+\left(2/7\right)^2+\left(-3/7\right)^2} = \sqrt{\frac{36}{49}+\frac{4}{49}+\frac{9}{49}} = \sqrt{\frac{49}{49}} = 1$$

Here, each of the given three vector is a unit vector.

$$\vec{a}.\vec{b}=\frac{2}{7}\times\frac{3}{7}+\frac{3}{7}\times\left(\frac{-6}{7}\right)+\frac{6}{7}\times\frac{2}{7}$$

$$=\frac{6}{49}+\left(\frac{-18}{49}\right)+\frac{12}{49}=\frac{6-18+12}{49}=0$$

$$\vec{b}.\vec{c}=\frac{3}{7}\times\frac{6}{7}+\left(\frac{-6}{7}\right)\times\frac{2}{7}+\frac{2}{7}\times\left(-\frac{3}{7}\right)$$

$$=\frac{18}{49}-\frac{12}{49}+\left(\frac{-6}{49}\right)=\frac{18-12-6}{49}=0$$

$$\vec{c}.\vec{a}=\frac{6}{7}\times\frac{2}{7}+\frac{2}{7}\times\frac{3}{7}+\left(-\frac{3}{7}\right)\times\frac{6}{7}$$

$$=\frac{12}{49}+\frac{6}{49}-\frac{18}{49}=0$$

Therefore, the given three vectors are mutually perpendicular to each other.

Q6. Find $|\vec{a}|$ and $|\vec{b}|$,if $(\vec{a}+\vec{b}).(\vec{a}-\vec{b})=8$ and $|\vec{a}|=8|\vec{b}|$

A.6. $|\vec{a}|$ **and** $|\vec{b}|$ **,if** $(\vec{a}+\vec{b}).(\vec{a}-\vec{b})=8$ **and** $|\vec{a}|=8|\vec{b}|$

$$(\vec{a}+\vec{b}).(\vec{a}-\vec{b})=8$$

$$and\ |\vec{a}|=8|\vec{b}|$$

$$(\vec{a}+\vec{b}).(\vec{a}-\vec{b})=8$$

$$\vec{a}.\vec{a}-\vec{a}.\vec{b}+\vec{b}.\vec{a}-\vec{b}.\vec{b}=8$$

$$|\vec{a}|^2-|\vec{b}|^2=8$$

$$(8|\vec{b}|)^2-|\vec{b}|^2=8$$

$$64|\vec{b}|^2-|\vec{b}|^2=8$$

$$63|\vec{b}|^2=8$$

$$|\vec{b}|=\sqrt{\frac{8}{63}}\ (\therefore magnitudeofavectorisnon-negative)$$

$$|\vec{b}|=\frac{2\sqrt{2}}{3\sqrt{7}}$$

$$And$$

$$|\vec{a}|=8|\vec{b}|=\frac{2\times2\sqrt{2}}{3\sqrt{7}}=\frac{16\sqrt{2}}{3\sqrt{7}}$$

Q7. Evaluate the product $(3\vec{a}-5\vec{b}).(2\vec{a}+7\vec{b})$.

A.7. $\left(3\vec{a}-5\vec{b}\right).\left(2\vec{a}+7\vec{b}\right)$.

$$=3\vec{a}.2\vec{a}+3\vec{a}.7\vec{b}-5\vec{b}.2\vec{a}-5\vec{b}.7\vec{b}$$
$$=6\vec{a}.\vec{a}+21\vec{a}.\vec{b}-10\vec{a}.\vec{b}-35\vec{b}.\vec{b}$$
$$=6\left|\vec{a}\right|^2+21\vec{a}.\vec{b}-10\vec{a}.\vec{b}-35\left|\vec{b}\right|^2$$
$$=6\left|\vec{a}\right|^2+11\vec{a}.\vec{b}-35\left|\vec{b}\right|^2$$

Q8. Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$ having the same magnitude such that the angle between them is 60° and their scalar product is 1/2.

A.8. Let θ be the angle between the vectors $\left|\vec{a}\right|$ and $\left|\vec{b}\right|$.

It is given that $\left|\vec{a}\right|=\left|\vec{b}\right|, \vec{a}.\vec{b}=\frac{1}{2} and \theta=60^\circ$ ----(1)

We know, $\vec{a}.\vec{b}=\left|\vec{a}\right|\left|\vec{b}\right|\cos\theta$

$$\therefore \frac{1}{2}=\left|\vec{a}\right|\left|\vec{a}\right|\cos 60^\circ (using (1))$$
$$\frac{1}{2}=\left|\vec{a}\right|^2\times\frac{1}{2}$$
$$\left|\vec{a}\right|^2=1$$
$$\left|\vec{a}\right|=\left|\vec{b}\right|=1$$

$\therefore$ Magnitude of two vector=1

Q9. Find $\left|\vec{x}\right|$ if for a unit vector $\left(\vec{x}-\vec{a}\right).\left(\vec{x}-\vec{a}\right)=12$.

A.9.

$$\left(\vec{x}-\vec{a}\right).\left(\vec{x}-\vec{a}\right)=12$$
$$\vec{x}.\vec{x}+\vec{x}.\vec{a}-\vec{a}.\vec{x}-\vec{a}.\vec{a}=12$$
$$\left|\vec{x}\right|^2-\left|\vec{a}\right|^2=12$$
$$\left|\vec{x}\right|^2-1=12\left[\left|\vec{a}\right|=1 \text{ as a } \vec{a} \text{ is unit vector}\right]$$
$$\left|\vec{x}\right|^2=12+1=13$$
$$\therefore \left|\vec{x}\right|=\sqrt{13}$$

Q10. If $\vec{a}=2\hat{i}+2\hat{j}+3\hat{k}, \vec{b}=-\hat{i}+1\vec{j}+\vec{k}$ **and** $c=3\hat{i}+\hat{j}$ **are such that** $\vec{a}+\lambda\vec{b}$ **is perpendicular to** $\vec{c}$ **then find the value of λ**

A.10. Given,

$$\vec{a}=2\hat{i}+2\hat{j}+3\hat{k}$$
$$\vec{b}=-\hat{i}+2\hat{j}+\hat{k}$$
$$\vec{c}=3\hat{i}+\hat{j}$$

Now,

$$\vec{a}+\lambda\vec{b}=\left(2\hat{i}+2\hat{j}+3\hat{k}\right)+\lambda\left(-\hat{i}+2\hat{j}+\hat{k}\right)$$
$$=\left(2\hat{i}+2\hat{j}+3\hat{k}\right)+\left(-\lambda\hat{i}+2\lambda\hat{j}+\lambda\hat{k}\right)$$
$$=(2-\lambda)\hat{i}+(2+2\lambda)\hat{j}+(3+\lambda)\hat{k}$$

If $\left(\vec{a}+\lambda\vec{b}\right)$ is perpendicular to $\vec{c}$, then $\left(\vec{a}+\lambda\vec{b}\right).\vec{c}=0$

$$=\left[(2-\lambda)\hat{i}+(2+2\lambda)\hat{j}+(3+\lambda)\hat{k}\right].\left(3\hat{i}+\hat{j}\right)$$
$$=3(2-\lambda)+1(2+2\lambda)+0(3+\lambda)$$
$$=6-3\lambda+2+2\lambda+0$$
$$=8-\lambda$$
$$\Rightarrow \lambda=8$$

Therefore, the required value of λ is 8.

Q11. Show that $|\vec{a}|\vec{b}+|\vec{b}|$ **is perpendicular to** $|\vec{a}|\vec{b}-|\vec{b}|\vec{a}$ **for any two non-zero vectors** $\vec{a}$ **and** $\vec{b}$

A.11. $\left(|\vec{a}|\vec{b}+|\vec{b}|\vec{a}\right).\left(|\vec{a}|\vec{b}-|\vec{b}|\vec{a}\right)$

$$=|\vec{a}|\vec{b}.|\vec{a}|\vec{b}-|\vec{a}|\vec{b}.|\vec{b}|\vec{a}+|\vec{b}|\vec{a}.|\vec{a}|\vec{b}-|\vec{b}|\vec{a}.|\vec{b}|\vec{a}$$
$$=|\vec{a}|^2\,\vec{b}.\vec{b}-|\vec{b}|^2\,\vec{a}.\vec{a}$$
$$=|\vec{a}|^2\,|\vec{b}|^2-|\vec{b}|^2\,|\vec{a}|^2$$
$$=0$$

$\therefore$ Therefore, $|\vec{a}|\vec{b}+|\vec{b}|\vec{a}$ and $|\vec{a}|\vec{b}-|\vec{b}|\vec{a}$ are perpendicular.

Q12. If and $\vec{a}.\vec{a}$ **= 0 and** $\vec{a}.\vec{b}$ **= 0 , then what can be concluded about the vector** $\vec{b}$ **?**

A.12. We know,

$\vec{a}.\vec{a}=0$ and $\vec{a}.\vec{b}=0$

Now,

$\vec{a}.\vec{a}=0 \Rightarrow |\vec{a}|^2 \Rightarrow |\vec{a}|=0$

$\therefore\ \vec{a}$ is a zero vector.

Thus, vector $\vec{b}$ satisfying $\vec{a}.\vec{b}=0$ can be any vector.

Q13. If $\vec{a},\vec{b}$ and $\vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}$ = 0 find the value of $\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$

A.13.

$$\left|\vec{a}+\vec{b}+\vec{c}\right| = \left(\vec{a}+\vec{b}+\vec{c}\right).\left(\vec{a}+\vec{b}+\vec{c}\right)$$

$$= \vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{a}.\vec{c}+\vec{b}.\vec{a}+\vec{b}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}+\vec{c}.\vec{b}+\vec{c}.\vec{c}$$

$$= |\vec{a}|^2+\left|\vec{b}\right|^2+|\vec{c}|^2+2\left(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}\right)$$

$$= 1+1+1+2\left(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}\right)$$

$$= 3+2\left(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}\right)$$

$$\Rightarrow \vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a} = \frac{-3}{2}$$

Q14. If either vector $\vec{a}=0$ or $\vec{b}=0$, then $\vec{a}.\vec{b}=0$. But the converse need not be true. Justify your answer with an example.

A14. Consider

$$\vec{a} = 2\hat{i}+4\hat{j}+3\hat{k}$$

and

$$\vec{b} = 3\hat{i}+3\hat{j}-6\hat{k}$$

Then,

$$\vec{a}.\vec{b} = 2.3+4.3+3.(-6)$$
$$= 6+12-18=0$$

$$|\vec{a}| = \sqrt{2^2+4^2+3^2} = \sqrt{29} \therefore a \neq 0$$

Now, $\left|\vec{b}\right| = \sqrt{3^2+3^2+(-6)^2} = \sqrt{9+9+36} = \sqrt{54} \therefore \vec{b} \neq 0$

Therefore, the converse of the given statement need not be true.

Q15. If the vertices A, B, C of a triangle ABC are (1, 2, 3), (–1, 0, 0), (0, 1, 2), respectively, then find ∠ABC. [∠ABC is the angle between the vectors $\overline{BA}$ and $\overline{BC}$]

A.15. Vertices of $\triangle ABC$ are given as

$$A(1,2,3), B(-1,0,0), C(0,1,2)$$

$\therefore \triangle ABC$ is the angle between the vectors $B\vec{A}$ and $B\vec{C}$

$$B\vec{A} = \{I - (-1)\}\hat{i} + (2-0)\hat{j} + (3-0)\hat{k}$$
$$= 2\hat{i} + 2\hat{j} + 3\hat{k}$$
$$B\vec{C} = \{0 - (-1)\}\hat{i} + (1-0)\hat{j} + (2-0)\hat{k}$$
$$= \hat{i} + \hat{j} + 2\hat{k}$$
$$\therefore B\vec{A}.B\vec{C} = \left(2\hat{i} + 2\hat{j} + 3\hat{k}\right).\left(\hat{i} + \hat{j} + 2\hat{k}\right)$$
$$= 2\times1 + 2\times1 + 3\times2$$
$$= 2 + 2 + 6$$
$$= 10$$
$$\left|B\vec{A}\right| = \sqrt{2^2 + 2^2 + 3^2} = \sqrt{4+4+9} = \sqrt{17}$$
$$\left|B\vec{C}\right| = \sqrt{1+1+2^2} = \sqrt{6}$$

Now, we know that

$$B\vec{A}.B\vec{C} = \left|B\vec{A}\right|.\left|B\vec{C}\right|.\cos(\angle ABC)$$
$$\therefore 10 = \sqrt{17}\times\sqrt{6}\cos(\angle ABC)$$
$$\cos(\angle ABC) = \frac{10}{\sqrt{17}\times\sqrt{6}}$$
$$\angle ABC = \cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$$

Here, the angle is $\cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$

Q16. Show that the points A (1, 2, 7), B (2, 6, 3) and C (3, 10, –1) are collinear.

A.16. Given, point are

$A(1,2,7), B(2,6,3), C(3,10,-1)$

$\vec{AB} = (2-1)\hat{i} + (6-2)\hat{j} + (3-7)\hat{k}$

$= \hat{i} + 4\hat{j} - 4\hat{k}$

$\vec{BC} = (3-2)\hat{i} + (10-6)\hat{j} + (-1-3)\hat{k}$

$= \hat{i} + 4\hat{j} - 4\hat{k}$

$\vec{AC} = (3-1)\hat{i} + (10-2)\hat{j} + (-1-7)\hat{k}$

$= 2\hat{i} + 8\hat{j} - 8\hat{k}$

Now,

$|\vec{AB}| = \sqrt{1^2 + 4^2 + (-4)^2} = \sqrt{1+16+16} = \sqrt{33}$

$|\vec{BC}| = \sqrt{1^2 + 4^2 + (-4)^2} = \sqrt{1+16+16} = \sqrt{33}$

$|\vec{AC}| = \sqrt{2^2 + 8^2 + 8^2} = \sqrt{4+64+64}$

$= \sqrt{132} = 2\sqrt{33}$

$\therefore |\vec{AC}| = |\vec{AB}| + |\vec{BC}|$

$2\sqrt{33} = \sqrt{33} + \sqrt{33}$

The given points are collinear.

Q17. Show that the vectors $2\hat{i} - \hat{j} + \hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$ form the vertices of a right angled triangle.

A.17. Let vector $2\hat{i} - \hat{j} + \hat{k}, \hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$ be position vector of point A, B, C respectively.

So,

$\vec{OA} = 2\hat{i} - \hat{j} + \hat{k}$

$\vec{OB} = \hat{i} - 3\hat{j} - 5\hat{k}$

$\vec{OC} = 3\hat{i} - 4\hat{j} - 4\hat{k}$

Now, vectors $\vec{AB}, \vec{BC}$ and $\vec{AC}$ represents the sides of $\triangle ABC$.

Hence,

$$\vec{AB} = (1-2)\hat{i} + (-3+1)\hat{j} + (-5-1)\hat{k}$$
$$= -\hat{i} - 2\hat{j} - 6\hat{k}$$
$$\vec{BC} = (3-1)\hat{i} + (-4+3)\hat{j} + (-4+5)\hat{k}$$
$$= 2\hat{i} - \hat{j} + \hat{k}$$
$$\vec{AC} = (2-3)\hat{i} + (-1+4)\hat{j} + (1+4)\hat{k}$$
$$= -\hat{i} + 3\hat{j} + 5\hat{k}$$
$$|\vec{AB}| = \sqrt{(-1)^2 + (-2)^2 + (-6)^2} = \sqrt{1+4+36} = \sqrt{41}$$
$$|\vec{BC}| = \sqrt{(2)^2 + (-1)^2 + 1} = \sqrt{4+1+1} = \sqrt{6}$$
$$|\vec{AC}| = \sqrt{(-1)^2 + (3)^2 + 5^2} = \sqrt{1+9+25} = \sqrt{35}$$
$$\therefore |\vec{BC}|^2 + |\vec{AC}|^2 = 6 + 35 = 41 = |\vec{AB}|^2$$

Therefore, $\triangle ABC$ is right angled triangle.

Q18. If $\vec{a}$ is a nonzero vector of magnitude 'a' and λ a nonzero scalar, then $\lambda\vec{a}$ is unit vector if

(A) λ = 1

(B) λ = –1
(C) a = | λ |
(D) a = 1/|λ|

A.18.Vector $\lambda\vec{a}$ is a unit vector if $|\lambda\vec{a}| = 1$

Now,

$$|\lambda\vec{a}| = 1$$
$$|\lambda||\vec{a}| = 1$$
$$|\vec{a}| = \frac{1}{|\lambda|}[\lambda \neq 0]$$
$$a = \frac{1}{|\lambda|}\left[|\vec{a}| = a\right]$$

$\therefore$ Therefore, vectar $\lambda\vec{a}$ is a unit vector if a=$\frac{1}{|\lambda|}$.

Option (D)is correct.