Miscellaneous Solutions

Question 1:

$$\frac{1}{x-x^3}$$

Answer

$$\frac{1}{x-x^3}=\frac{1}{x\left(1-x^2\right)}=\frac{1}{x(1-x)(1+x)}$$

$$\text{Let } \frac{1}{x(1-x)(1+x)}=\frac{A}{x}+\frac{B}{(1-x)}+\frac{C}{1+x} \qquad \ldots(1)$$

$$\Rightarrow 1=A\left(1-x^2\right)+Bx(1+x)+Cx(1-x)$$

$$\Rightarrow 1=A-Ax^2+Bx+Bx^2+Cx-Cx^2$$

Equating the coefficients of x^2, x, and constant term, we obtain

$-A + B - C = 0$

$B + C = 0$

$A = 1$

On solving these equations, we obtain

$$A=1, B=\frac{1}{2}, \text{ and } C=-\frac{1}{2}$$

From equation (1), we obtain

$$\frac{1}{x(1-x)(1+x)}=\frac{1}{x}+\frac{1}{2(1-x)}-\frac{1}{2(1+x)}$$

$$\Rightarrow \int \frac{1}{x(1-x)(1+x)}dx=\int \frac{1}{x}dx+\frac{1}{2}\int \frac{1}{1-x}dx-\frac{1}{2}\int \frac{1}{1+x}dx$$

$$=\log|x|-\frac{1}{2}\log|(1-x)|-\frac{1}{2}\log|(1+x)|$$

$$=\log|x|-\log\left|(1-x)^{\frac{1}{2}}\right|-\log\left|(1+x)^{\frac{1}{2}}\right|$$

$$=\log\left|\frac{x}{(1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}}\right|+C$$

$$=\log\left|\left(\frac{x^2}{1-x^2}\right)^{\frac{1}{2}}\right|+C$$

$$=\frac{1}{2}\log\left|\frac{x^2}{1-x^2}\right|+C$$

Question 2:

$$\frac{1}{\sqrt{x+a}+\sqrt{(x+b)}}$$

Answer

$$\frac{1}{\sqrt{x+a}+\sqrt{x+b}}=\frac{1}{\sqrt{x+a}+\sqrt{x+b}}\times\frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}}$$

$$=\frac{\sqrt{x+a}-\sqrt{x+b}}{(x+a)-(x+b)}$$

$$=\frac{\left(\sqrt{x+a}-\sqrt{x+b}\right)}{a-b}$$

$$\Rightarrow\int\frac{1}{\sqrt{x+a}-\sqrt{x+b}}dx=\frac{1}{a-b}\int\left(\sqrt{x+a}-\sqrt{x+b}\right)dx$$

$$=\frac{1}{(a-b)}\left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}}-\frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}}\right]$$

$$=\frac{2}{3(a-b)}\left[(x+a)^{\frac{3}{2}}-(x+b)^{\frac{3}{2}}\right]+\text{C}$$

Question 3:

$\frac{1}{x\sqrt{ax-x^2}}$ [Hint: Put $x=\frac{a}{t}$]

Answer

$$\frac{1}{x\sqrt{ax-x^2}}$$

Let $x=\frac{a}{t} \Rightarrow dx=-\frac{a}{t^2}dt$

$$\Rightarrow \int \frac{1}{x\sqrt{ax-x^2}}dx=\int \frac{1}{\frac{a}{t}\sqrt{a\cdot\frac{a}{t}-\left(\frac{a}{t}\right)^2}}\left(-\frac{a}{t^2}dt\right)$$

$$=-\int \frac{1}{at}\cdot\frac{1}{\sqrt{\frac{1}{t}-\frac{1}{t^2}}}dt$$

$$=-\frac{1}{a}\int \frac{1}{\sqrt{\frac{t^2}{t}-\frac{t^2}{t^2}}}dt$$

$$=-\frac{1}{a}\int \frac{1}{\sqrt{t-1}}dt$$

$$=-\frac{1}{a}\left[2\sqrt{t-1}\right]+C$$

$$=-\frac{1}{a}\left[2\sqrt{\frac{a}{x}-1}\right]+C$$

$$=-\frac{2}{a}\left(\frac{\sqrt{a-x}}{\sqrt{x}}\right)+C$$

$$=-\frac{2}{a}\left(\sqrt{\frac{a-x}{x}}\right)+C$$

Question 4:

$$\frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}$$

Answer

$$\frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}$$

Multiplying and dividing by x^{-3}, we obtain

$$\frac{x^{-3}}{x^2\cdot x^{-3}\left(x^4+1\right)^{\frac{3}{4}}}=\frac{x^{-3}\left(x^4+1\right)^{\frac{-3}{4}}}{x^2\cdot x^{-3}}$$

$$=\frac{\left(x^4+1\right)^{\frac{-3}{4}}}{x^5\cdot\left(x^4\right)^{-\frac{3}{4}}}$$

$$=\frac{1}{x^5}\left(\frac{x^4+1}{x^4}\right)^{-\frac{3}{4}}$$

$$=\frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{-\frac{3}{4}}$$

Let $\frac{1}{x^4}=t \Rightarrow -\frac{4}{x^5}dx=dt \Rightarrow \frac{1}{x^5}dx=-\frac{dt}{4}$

$$\therefore \int\frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}dx=\int\frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{-\frac{3}{4}}dx$$

$$=-\frac{1}{4}\int(1+t)^{-\frac{3}{4}}dt$$

$$=-\frac{1}{4}\left[\frac{(1+t)^{\frac{1}{4}}}{\frac{1}{4}}\right]+C$$

$$=-\frac{1}{4}\frac{\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}}{\frac{1}{4}}+C$$

$$=-\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}+C$$

Question 5:

$$\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}\left[\text{Hint: } \frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}=\frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)} \text{ Put } x=t^{6}\right]$$

Answer

$$\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}=\frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}$$

$$\text{Let } x=t^{6} \Rightarrow dx=6t^{5}dt$$

$$\therefore \int \frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}dx=\int \frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}dx$$

$$=\int \frac{6t^{5}}{t^{2}(1+t)}dt$$

$$=6\int \frac{t^{3}}{(1+t)}dt$$

On dividing, we obtain

$$\int \frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}dx=6\int\left\{\left(t^{2}-t+1\right)-\frac{1}{1+t}\right\}dt$$

$$=6\left[\left(\frac{t^{3}}{3}\right)-\left(\frac{t^{2}}{2}\right)+t-\log|1+t|\right]$$

$$=2x^{\frac{1}{2}}-3x^{\frac{1}{3}}+6x^{\frac{1}{6}}-6\log\left(1+x^{\frac{1}{6}}\right)+C$$

$$=2\sqrt{x}-3x^{\frac{1}{3}}+6x^{\frac{1}{6}}-6\log\left(1+x^{\frac{1}{6}}\right)+C$$

Question 6:

$$\frac{5x}{(x+1)\left(x^{2}+9\right)}$$

Answer

Let $\frac{5x}{(x+1)(x^2+9)}=\frac{A}{(x+1)}+\frac{Bx+C}{(x^2+9)}$ ...(1)

$\Rightarrow 5x=A(x^2+9)+(Bx+C)(x+1)$

$\Rightarrow 5x=Ax^2+9A+Bx^2+Bx+Cx+C$

Equating the coefficients of x^2, x, and constant term, we obtain

$A + B = 0$

$B + C = 5$

$9A + C = 0$

On solving these equations, we obtain

$A=-\frac{1}{2}, B=\frac{1}{2}, \text{ and } C=\frac{9}{2}$

From equation (1), we obtain

$$\frac{5x}{(x+1)(x^2+9)}=\frac{-1}{2(x+1)}+\frac{\frac{x}{2}+\frac{9}{2}}{(x^2+9)}$$

$$\int\frac{5x}{(x+1)(x^2+9)}dx=\int\left\{\frac{-1}{2(x+1)}+\frac{(x+9)}{2(x^2+9)}\right\}dx$$

$$=-\frac{1}{2}\log|x+1|+\frac{1}{2}\int\frac{x}{x^2+9}dx+\frac{9}{2}\int\frac{1}{x^2+9}dx$$

$$=-\frac{1}{2}\log|x+1|+\frac{1}{4}\int\frac{2x}{x^2+9}dx+\frac{9}{2}\int\frac{1}{x^2+9}dx$$

$$=-\frac{1}{2}\log|x+1|+\frac{1}{4}\log|x^2+9|+\frac{9}{2}\cdot\frac{1}{3}\tan^{-1}\frac{x}{3}$$

$$=-\frac{1}{2}\log|x+1|+\frac{1}{4}\log(x^2+9)+\frac{3}{2}\tan^{-1}\frac{x}{3}+C$$

Question 7:

$\frac{\sin x}{\sin(x-a)}$

Answer

$$\frac{\sin x}{\sin(x-a)}$$

Let $x - a = t$ □ $dx = dt$

$$\int \frac{\sin x}{\sin(x-a)}dx = \int \frac{\sin(t+a)}{\sin t}dt$$

$$= \int \frac{\sin t \cos a + \cos t \sin a}{\sin t}dt$$

$$= \int (\cos a + \cot t \sin a)dt$$

$$= t\cos a + \sin a \log|\sin t| + C_1$$

$$= (x-a)\cos a + \sin a \log|\sin(x-a)| + C_1$$

$$= x\cos a + \sin a \log|\sin(x-a)| - a\cos a + C_1$$

$$= \sin a \log|\sin(x-a)| + x\cos a + C$$

Question 8:

$$\frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}$$

Answer

$$\frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}} = \frac{e^{4\log x}\left(e^{\log x} - 1\right)}{e^{2\log x}\left(e^{\log x} - 1\right)}$$

$$= e^{2\log x}$$

$$= e^{\log x^2}$$

$$= x^2$$

$$\therefore \int \frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}dx = \int x^2 dx = \frac{x^3}{3} + C$$

Question 9:

$$\frac{\cos x}{\sqrt{4-\sin^2 x}}$$

Answer

$$\frac{\cos x}{\sqrt{4-\sin^2 x}}$$

Let sin x = t □ cos x dx = dt

$$\Rightarrow \int \frac{\cos x}{\sqrt{4-\sin^2 x}} dx = \int \frac{dt}{\sqrt{(2)^2-(t)^2}}$$

$$= \sin^{-1}\left(\frac{t}{2}\right)+C$$

$$= \sin^{-1}\left(\frac{\sin x}{2}\right)+C$$

Question 10:

$$\frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}$$

Answer

$$\frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x} = \frac{\left(\sin^4 x+\cos^4 x\right)\left(\sin^4 x-\cos^4 x\right)}{\sin^2 x+\cos^2 x-\sin^2 x\cos^2 x-\sin^2 x\cos^2 x}$$

$$= \frac{\left(\sin^4 x+\cos^4 x\right)\left(\sin^2 x+\cos^2 x\right)\left(\sin^2 x-\cos^2 x\right)}{\left(\sin^2 x-\sin^2 x\cos^2 x\right)+\left(\cos^2 x-\sin^2 x\cos^2 x\right)}$$

$$= \frac{\left(\sin^4 x+\cos^4 x\right)\left(\sin^2 x-\cos^2 x\right)}{\sin^2 x\left(1-\cos^2 x\right)+\cos^2 x\left(1-\sin^2 x\right)}$$

$$= \frac{-\left(\sin^4 x+\cos^4 x\right)\left(\cos^2 x-\sin^2 x\right)}{\left(\sin^4 x+\cos^4 x\right)}$$

$$= -\cos 2x$$

$$\therefore \int \frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x} dx = \int -\cos 2x\, dx = -\frac{\sin 2x}{2}+C$$

Question 11:

$$\frac{1}{\cos(x+a)\cos(x+b)}$$

Answer

$$\frac{1}{\cos(x+a)\cos(x+b)}$$

Multiplying and dividing by $\sin(a-b)$, we obtain

$$\frac{1}{\sin(a-b)}\left[\frac{\sin(a-b)}{\cos(x+a)\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\frac{\sin[(x+a)-(x+b)]}{\cos(x+a)\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\frac{\sin(x+a)\cdot\cos(x+b)-\cos(x+a)\sin(x+b)}{\cos(x+a)\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\frac{\sin(x+a)}{\cos(x+a)}-\frac{\sin(x+b)}{\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\tan(x+a)-\tan(x+b)\right]$$

$$\int\frac{1}{\cos(x+a)\cos(x+b)}dx=\frac{1}{\sin(a-b)}\int\left[\tan(x+a)-\tan(x+b)\right]dx$$

$$=\frac{1}{\sin(a-b)}\left[-\log\left|\cos(x+a)\right|+\log\left|\cos(x+b)\right|\right]+C$$

$$=\frac{1}{\sin(a-b)}\log\left|\frac{\cos(x+b)}{\cos(x+a)}\right|+C$$

Question 12:

$$\frac{x^3}{\sqrt{1-x^8}}$$

Answer

$$\frac{x^3}{\sqrt{1-x^8}}$$

Let $x^4 = t$ □ $4x^3\, dx = dt$

$$\Rightarrow \int \frac{x^3}{\sqrt{1-x^8}}dx = \frac{1}{4}\int \frac{dt}{\sqrt{1-t^2}}$$

$$= \frac{1}{4}\sin^{-1} t + C$$

$$= \frac{1}{4}\sin^{-1}\left(x^4\right) + C$$

Question 13:

$$\frac{e^x}{\left(1+e^x\right)\left(2+e^x\right)}$$

Answer

$$\frac{e^x}{\left(1+e^x\right)\left(2+e^x\right)}$$

Let $e^x = t$ □ $e^x\, dx = dt$

$$\Rightarrow \int \frac{e^x}{\left(1+e^x\right)\left(2+e^x\right)}dx = \int \frac{dt}{(t+1)(t+2)}$$

$$= \int\left[\frac{1}{(t+1)} - \frac{1}{(t+2)}\right]dt$$

$$= \log|t+1| - \log|t+2| + C$$

$$= \log\left|\frac{t+1}{t+2}\right| + C$$

$$= \log\left|\frac{1+e^x}{2+e^x}\right| + C$$

Question 14:

$$\frac{1}{\left(x^2+1\right)\left(x^2+4\right)}$$

Answer

$$\therefore \frac{1}{(x^2+1)(x^2+4)} = \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+4)}$$

$$\Rightarrow 1 = (Ax+B)(x^2+4) + (Cx+D)(x^2+1)$$

$$\Rightarrow 1 = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + Cx + Dx^2 + D$$

Equating the coefficients of x^3, x^2, x, and constant term, we obtain

$A + C = 0$

$B + D = 0$

$4A + C = 0$

$4B + D = 1$

On solving these equations, we obtain

$$A = 0,\ B = \frac{1}{3},\ C = 0, \text{and } D = -\frac{1}{3}$$

From equation (1), we obtain

$$\frac{1}{(x^2+1)(x^2+4)} = \frac{1}{3(x^2+1)} - \frac{1}{3(x^2+4)}$$

$$\int \frac{1}{(x^2+1)(x^2+4)} dx = \frac{1}{3}\int \frac{1}{x^2+1} dx - \frac{1}{3}\int \frac{1}{x^2+4} dx$$

$$= \frac{1}{3}\tan^{-1} x - \frac{1}{3}\cdot\frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

$$= \frac{1}{3}\tan^{-1} x - \frac{1}{6}\tan^{-1}\frac{x}{2} + C$$

Question 15:

$\cos^3 x e^{\log \sin x}$

Answer

$\cos^3 x e^{\log \sin x}$ = $\cos^3 x \times \sin x$

Let $\cos x = t$ □ $-\sin x\, dx = dt$

$$\Rightarrow \int \cos^3 x\, e^{\log \sin x} dx = \int \cos^3 x \sin x dx$$
$$= -\int t \cdot dt$$
$$= -\frac{t^4}{4} + C$$
$$= -\frac{\cos^4 x}{4} + C$$

Question 16:

$e^{3\log x}\left(x^4+1\right)^{-1}$

Answer

$$e^{3\log x}\left(x^4+1\right)^{-1} = e^{\log x^3}\left(x^4+1\right)^{-1} = \frac{x^3}{\left(x^4+1\right)}$$

Let $x^4 + 1 = t \Rightarrow 4x^3\, dx = dt$

$$\Rightarrow \int e^{3\log x}\left(x^4+1\right)^{-1} dx = \int \frac{x^3}{\left(x^4+1\right)} dx$$
$$= \frac{1}{4}\int \frac{dt}{t}$$
$$= \frac{1}{4}\log|t| + C$$
$$= \frac{1}{4}\log\left|x^4+1\right| + C$$
$$= \frac{1}{4}\log\left(x^4+1\right) + C$$

Question 17:

$f'(ax+b)\left[f(ax+b)\right]^n$

Answer

$$f'(ax+b)\left[f(ax+b)\right]^n$$

Let $f(ax+b)=t \Rightarrow af'(ax+b)dx=dt$

$$\Rightarrow \int f'(ax+b)\left[f(ax+b)\right]^n dx=\frac{1}{a}\int t^n dt$$

$$=\frac{1}{a}\left[\frac{t^{n+1}}{n+1}\right]$$

$$=\frac{1}{a(n+1)}\left(f(ax+b)\right)^{n+1}+C$$

Question 18:

$$\frac{1}{\sqrt{\sin^3 x\sin(x+\alpha)}}$$

Answer

$$\frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}} = \frac{1}{\sqrt{\sin^3 x(\sin x \cos\alpha + \cos x \sin\alpha)}}$$

$$= \frac{1}{\sqrt{\sin^4 x \cos\alpha + \sin^3 x \cos x \sin\alpha}}$$

$$= \frac{1}{\sin^2 x\sqrt{\cos\alpha + \cot x \sin\alpha}}$$

$$= \frac{\operatorname{cosec}^2 x}{\sqrt{\cos\alpha + \cot x \sin\alpha}}$$

Let $\cos\alpha + \cot x \sin\alpha = t \Rightarrow -\operatorname{cosec}^2 x \sin\alpha \, dx = dt$

$$\therefore \int \frac{1}{\sin^3 x \sin(x+\alpha)} dx = \int \frac{\operatorname{cosec}^2 x}{\sqrt{\cos\alpha + \cot x \sin\alpha}} dx$$

$$= \frac{-1}{\sin\alpha} \int \frac{dt}{\sqrt{t}}$$

$$= \frac{-1}{\sin\alpha}\left[2\sqrt{t}\right] + C$$

$$= \frac{-1}{\sin\alpha}\left[2\sqrt{\cos\alpha + \cot x \sin\alpha}\right] + C$$

$$= \frac{-2}{\sin\alpha}\sqrt{\cos\alpha + \frac{\cos x \sin\alpha}{\sin x}} + C$$

$$= \frac{-2}{\sin\alpha}\sqrt{\frac{\sin x \cos\alpha + \cos x \sin\alpha}{\sin x}} + C$$

$$= -\frac{2}{\sin\alpha}\sqrt{\frac{\sin(x+\alpha)}{\sin x}} + C$$

Question 19:

$$\frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}, x \in [0,1]$$

Answer

Let $I=\int \frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}dx$

It is known that, $\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}=\frac{\pi}{2}$

$$\Rightarrow I=\int \frac{\left(\frac{\pi}{2}-\cos^{-1}\sqrt{x}\right)-\cos^{-1}\sqrt{x}}{\frac{\pi}{2}}dx$$

$$=\frac{2}{\pi}\int\left(\frac{\pi}{2}-2\cos^{-1}\sqrt{x}\right)dx$$

$$=\frac{2}{\pi}\cdot\frac{\pi}{2}\int 1\cdot dx-\frac{4}{\pi}\int\cos^{-1}\sqrt{x}\,dx$$

$$=x-\frac{4}{\pi}\int\cos^{-1}\sqrt{x}\,dx \qquad \ldots(1)$$

Let $I_1=\int\cos^{-1}\sqrt{x}\,dx$

Also, let $\sqrt{x}=t \;\Rightarrow\; dx=2t\,dt$

$$\Rightarrow I_1=2\int\cos^{-1}t\cdot t\,dt$$

$$=2\left[\cos^{-1}t\cdot\frac{t^2}{2}-\int\frac{-1}{\sqrt{1-t^2}}\cdot\frac{t^2}{2}dt\right]$$

$$=t^2\cos^{-1}t+\int\frac{t^2}{\sqrt{1-t^2}}dt$$

$$=t^2\cos^{-1}t-\int\frac{1-t^2-1}{\sqrt{1-t^2}}dt$$

$$=t^2\cos^{-1}t-\int\sqrt{1-t^2}\,dt+\int\frac{1}{\sqrt{1-t^2}}dt$$

$$=t^2\cos^{-1}t-\frac{t}{2}\sqrt{1-t^2}-\frac{1}{2}\sin^{-1}t+\sin^{-1}t$$

$$=t^2\cos^{-1}t-\frac{t}{2}\sqrt{1-t^2}+\frac{1}{2}\sin^{-1}t$$

From equation (1), we obtain

$$I = x - \frac{4}{\pi}\left[t^2 \cos t - \frac{t}{2}\sqrt{1-t^2} + \frac{1}{2}\sin^{-1} t\right]$$

$$= x - \frac{4}{\pi}\left[x\cos^{-1}\sqrt{x} - \frac{\sqrt{x}}{2}\sqrt{1-x} + \frac{1}{2}\sin^{-1}\sqrt{x}\right]$$

$$= x - \frac{4}{\pi}\left[x\left(\frac{}{2} - \sin^{-1}\sqrt{x}\right) - \frac{\sqrt{x-x^2}}{2} + \frac{}{2}\sin^{-1}\sqrt{x}\right]$$

$$= x - 2x + \frac{4x}{\pi}\sin^{-1}\sqrt{x} + \frac{2}{\pi}\sqrt{x-x^2} - \frac{2}{\pi}\sin^{-1}\sqrt{x}$$

$$= -x + \frac{2}{\pi}\left[(2x-1)\sin^{-1}\sqrt{x}\right] + \frac{2}{\pi}\sqrt{x-x^2} + C$$

$$= \frac{2(2x-1)}{\pi}\sin^{-1}\sqrt{x} + \frac{2}{\pi}\sqrt{x-x^2} - x + C$$

Question 20:

$$\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$$

Answer

$$I = \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx$$

$$\text{Let } x = \cos^2\theta \Rightarrow dx = -2\sin\theta\cos\theta\, d\theta$$

$$I = \int\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}(-2\sin\theta\cos\theta)\, d\theta$$

$$= -\int\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\sin 2\theta\, d\theta$$

$$= -\int\tan\frac{\theta}{2}\cdot 2\sin\theta\cos\theta\, d\theta$$

$$= -2\int\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\cos\theta\, d\theta$$

$$= -4\int \sin^2\frac{\theta}{2}\cos\theta\, d\theta$$

$$= -4\int \sin^2\frac{\theta}{2}\cdot\left(2\cos^2\frac{\theta}{2}-1\right)d\theta$$

$$= -4\int\left(2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right)d\theta$$

$$= -8\int \sin^2\frac{\theta}{2}\cdot\cos^2\frac{\theta}{2}d\theta+4\int\sin^2\frac{\theta}{2}d\theta$$

$$= -2\int \sin^2\theta\, d\theta+4\int\sin^2\frac{\theta}{2}d\theta$$

$$= -2\int\left(\frac{1-\cos 2\theta}{2}\right)d\theta+4\int\frac{1-\cos\theta}{2}d\theta$$

$$= -2\left[\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right]+4\left[\frac{\theta}{2}-\frac{\sin\theta}{2}\right]+C$$

$$= -\theta+\frac{\sin 2\theta}{2}+2\theta-2\sin\theta+C$$

$$= \theta+\frac{\sin 2\theta}{2}-2\sin\theta+C$$

$$= \theta+\frac{2\sin\theta\cos\theta}{2}-2\sin\theta+C$$

$$= \theta+\sqrt{1-\cos^2\theta}\cdot\cos\theta-2\sqrt{1-\cos^2\theta}+C$$

$$= \cos^{-1}\sqrt{x}+\sqrt{1-x}\cdot\sqrt{x}-2\sqrt{1-x}+C$$

$$= -2\sqrt{1-x}+\cos^{-1}\sqrt{x}+\sqrt{x(1-x)}+C$$

$$= -2\sqrt{1-x}+\cos^{-1}\sqrt{x}+\sqrt{x-x^2}+C$$

Question 21:

$$\frac{2+\sin 2x}{1+\cos 2x}e^x$$

Answer

$$I = \int\left(\frac{2+\sin 2x}{1+\cos 2x}\right)e^x$$
$$= \int\left(\frac{2+2\sin x\cos x}{2\cos^2 x}\right)e^x$$
$$= \int\left(\frac{1+\sin x\cos x}{\cos^2 x}\right)e^x$$
$$= \int\left(\sec^2 x+\tan x\right)e^x$$

Let $f(x)=\tan x \Rightarrow f'(x)=\sec^2 x$

$\therefore I = \int\left(f(x)+f'(x)\right]e^x dx$

$= e^x f(x)+C$

$= e^x \tan x+C$

Question 22:

$$\frac{x^2+x+1}{(x+1)^2(x+2)}$$

Answer

Let $\dfrac{x^2+x+1}{(x+1)^2(x+2)} = \dfrac{A}{(x+1)}+\dfrac{B}{(x+1)^2}+\dfrac{C}{(x+2)}$ ...(1)

$\Rightarrow x^2+x+1 = A(x+1)(x+2)+B(x+2)+C(x^2+2x+1)$

$\Rightarrow x^2+x+1 = A(x^2+3x+2)+B(x+2)+C(x^2+2x+1)$

$\Rightarrow x^2+x+1 = (A+C)x^2+(3A+B+2C)x+(2A+2B+C)$

Equating the coefficients of x^2, x,and constant term, we obtain

$A + C = 1$

$3A + B + 2C = 1$

$2A + 2B + C = 1$

On solving these equations, we obtain

$A = -2$, $B = 1$, and $C = 3$

From equation (1), we obtain

$$\frac{x^2+x+1}{(x+1)^2(x+2)}=\frac{-2}{(x+1)}+\frac{3}{(x+2)}+\frac{1}{(x+1)^2}$$

$$\int\frac{x^2+x+1}{(x+1)^2(x+2)}dx=-2\int\frac{1}{x+1}dx+3\int\frac{1}{(x+2)}dx+\int\frac{1}{(x+1)^2}dx$$

$$=-2\log|x+1|+3\log|x+2|-\frac{1}{(x+1)}+C$$

Question 23:

$$\tan^{-1}\sqrt{\frac{1-x}{1+x}}$$

Answer

$$I=\tan^{-1}\sqrt{\frac{1-x}{1+x}}dx$$

$$\text{Let } x=\cos\theta \Rightarrow dx=-\sin\theta d\theta$$

$$I=\int\tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}(-\sin\theta d\theta)$$

$$=-\int\tan^{-1}\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\sin\theta d\theta$$

$$=-\int\tan^{-1}\tan\frac{\theta}{2}\cdot\sin\theta d\theta$$

$$=-\frac{1}{2}\int\theta\cdot\sin\theta d\theta$$

$$=-\frac{1}{2}\left[\theta\cdot(-\cos\theta)-\int1\cdot(-\cos\theta)d\theta\right]$$

$$=-\frac{1}{2}[-\theta\cos\theta+\sin\theta]$$

$$=+\frac{1}{2}\theta\cos\theta-\frac{1}{2}\sin\theta$$

$$=\frac{1}{2}\cos^{-1}x\cdot x-\frac{1}{2}\sqrt{1-x^2}+C$$

$$=\frac{x}{2}\cos^{-1}x-\frac{1}{2}\sqrt{1-x^2}+C$$

$$=\frac{1}{2}\left(x\cos^{-1}x-\sqrt{1-x^2}\right)+C$$

Question 24:

$$\frac{\sqrt{x^2+1}\left[\log\left(x^2+1\right)-2\log x\right]}{x^4}$$

Answer

$$\frac{\sqrt{x^2+1}\left[\log\left(x^2+1\right)-2\log x\right]}{x^4}=\frac{\sqrt{x^2+1}}{x^4}\left[\log\left(x^2+1\right)-\log x^2\right]$$

$$=\frac{\sqrt{x^2+1}}{x^4}\left[\log\left(\frac{x^2+1}{x^2}\right)\right]$$

$$=\frac{\sqrt{x^2+1}}{x^4}\log\left(1+\frac{1}{x^2}\right)$$

$$=\frac{1}{x^3}\sqrt{\frac{x^2+1}{x^2}}\log\left(1+\frac{1}{x^2}\right)$$

$$=\frac{1}{x^3}\sqrt{1+\frac{1}{x^2}}\log\left(1+\frac{1}{x^2}\right)$$

$$\text{Let } 1+\frac{1}{x^2}=t \Rightarrow \frac{-2}{x^3}dx=dt$$

$$\therefore I=\int\frac{1}{x^3}\sqrt{1+\frac{1}{x^2}}\log\left(1+\frac{1}{x^2}\right)dx$$

$$=-\frac{1}{2}\int\sqrt{t}\log t\,dt$$

$$=-\frac{1}{2}\int t^{\frac{1}{2}}\cdot\log t\,dt$$

Integrating by parts, we obtain

$$I=-\frac{1}{2}\left[\log t\cdot\int t^{\frac{1}{2}}dt-\left\{\left(\frac{d}{dt}\log t\right)\int t^{\frac{1}{2}}dt\right\}dt\right]$$

$$=-\frac{1}{2}\left[\log t\cdot\frac{t^{\frac{3}{2}}}{\frac{3}{2}}-\int\frac{1}{t}\cdot\frac{t^{\frac{3}{2}}}{\frac{3}{2}}dt\right]$$

$$=-\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t-\frac{2}{3}\int t^{\frac{1}{2}}dt\right]$$

$$=-\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t-\frac{4}{9}t^{\frac{3}{2}}\right]$$

$$=-\frac{1}{3}t^{\frac{3}{2}}\log t+\frac{2}{9}t^{\frac{3}{2}}$$

$$=-\frac{1}{3}t^{\frac{3}{2}}\left[\log t-\frac{2}{3}\right]$$

$$=-\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{\frac{3}{2}}\left[\log\left(1+\frac{1}{x^2}\right)-\frac{2}{3}\right]+C$$

Question 25:

$$\int_{\frac{\pi}{2}}^{\pi}e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$$

Answer

$$I=\int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$$

$$=\int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right)dx$$

$$=\int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{\operatorname{cosec}^2\frac{x}{2}}{2}-\cot\frac{x}{2}\right)dx$$

$$\text{Let } f(x)=-\cot\frac{x}{2}$$

$$\Rightarrow f'(x)=-\left(-\frac{1}{2}\operatorname{cosec}^2\frac{x}{2}\right)=\frac{1}{2}\operatorname{cosec}^2\frac{x}{2}$$

$$\therefore I=\int_{\frac{\pi}{2}}^{\pi} e^x\left(f(x)+f'(x)\right]dx$$

$$=\left[e^x\cdot f(x)dx\right]_{\frac{\pi}{2}}^{\pi}$$

$$=-\left[e^x\cdot\cot\frac{x}{2}\right]_{\frac{\pi}{2}}^{\pi}$$

$$=-\left[e^{\pi}\times\cot\frac{\pi}{2}-e^{\frac{\pi}{2}}\times\cot\frac{\pi}{4}\right]$$

$$=-\left[e^{\pi}\times 0-e^{\frac{\pi}{2}}\times 1\right]$$

$$=e^{\frac{\pi}{2}}$$

Question 26:

$$\int_0^{\frac{\pi}{4}}\frac{\sin x\cos x}{\cos^4 x+\sin^4 x}dx$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\frac{(\sin x \cos x)}{\cos^4 x}}{\frac{(\cos^4 x + \sin^4 x)}{\cos^4 x}} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\tan x \sec^2 x}{1 + \tan^4 x} dx$$

$$\text{Let } \tan^2 x = t \ \Rightarrow \ 2\tan x \sec^2 x \, dx = dt$$

When $x = 0$, $t = 0$ and $\text{when } x = \frac{\pi}{4}, t = 1$

$$\therefore I = \frac{1}{2}\int_0^1 \frac{dt}{1+t^2}$$

$$= \frac{1}{2}\left[\tan^{-1} t\right]_0^1$$

$$= \frac{1}{2}\left[\tan^{-1} 1 - \tan^{-1} 0\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4}\right]$$

$$= \frac{\pi}{8}$$

Question 27:

$$\int_0^{\frac{\pi}{2}} \frac{\cos^2 x \, dx}{\cos^2 x + 4\sin^2 x}$$

Answer

Let $I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4\sin^2 x} dx$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4\left(1 - \cos^2 x\right)} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4 - 4\cos^2 x} dx$$

$$\Rightarrow I = \frac{-1}{3} \int_0^{\frac{\pi}{2}} \frac{4 - 3\cos^2 x - 4}{4 - 3\cos^2 x} dx$$

$$\Rightarrow I = \frac{-1}{3} \int_0^{\frac{\pi}{2}} \frac{4 - 3\cos^2 x}{4 - 3\cos^2 x} dx + \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{4}{4 - 3\cos^2 x} dx$$

$$\Rightarrow I = \frac{-1}{3} \int_0^{\frac{\pi}{2}} 1\, dx + \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{4\sec^2 x}{4\sec^2 x - 3} dx$$

$$\Rightarrow I = \frac{-1}{3} [x]_0^{\frac{\pi}{2}} + \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{4\sec^2 x}{4\left(1 + \tan^2 x\right) - 3} dx$$

$$\Rightarrow I = -\frac{\pi}{6} + \frac{2}{3} \int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1 + 4\tan^2 x} dx \qquad \ldots(1)$$

Consider, $\int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1 + 4\tan^2 x} dx$

Let $2\tan x = t \Rightarrow 2\sec^2 x\, dx = dt$

When $x = 0, t = 0$ and when $x = \frac{\pi}{2}, t = \infty$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1 + 4\tan^2 x} dx = \int_0^{\infty} \frac{dt}{1 + t^2}$$

$$= \left[\tan^{-1} t\right]_0^{\infty}$$

$$= \left[\tan^{-1}(\infty) - \tan^{-1}(0)\right]$$

$$= \frac{\pi}{2}$$

Therefore, from (1), we obtain

$$I = -\frac{\pi}{6} + \frac{2}{3}\left[\frac{\pi}{2}\right] = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

Question 28:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$$

Answer

$$\text{Let } I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$$

$$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x + \cos x)}{\sqrt{-(-\sin 2x)}} dx$$

$$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{-(-1+1-2\sin x \cos x)}} dx$$

$$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x + \cos x)}{\sqrt{1-(\sin^2 x + \cos^2 x - 2\sin x \cos x)}} dx$$

$$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x + \cos x)\,dx}{\sqrt{1-(\sin x - \cos x)^2}}$$

$$\text{Let } (\sin x - \cos x) = t \ \Rightarrow \ (\sin x + \cos x)\,dx = dt$$

When $x = \frac{\pi}{6}, t = \left(\frac{1-\sqrt{3}}{2}\right)$ and when $x = \frac{\pi}{3}, t = \left(\frac{\sqrt{3}-1}{2}\right)$

$$I = \int_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

$$\Rightarrow I = \int_{-\left(\frac{\sqrt{3}-1}{2}\right)}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

As $\frac{1}{\sqrt{1-(-t)^2}} = \frac{1}{\sqrt{1-t^2}}$, therefore, $\frac{1}{\sqrt{1-t^2}}$ is an even function.

It is known that if $f(x)$ is an even function, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

$$\Rightarrow I = 2\int_0^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

$$= \left[2\sin^{-1} t\right]_0^{\frac{\sqrt{3}-1}{2}}$$

$$= 2\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$$

Question 29:

$$\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$$

Answer

$$\text{Let } I = \int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$$

$$I = \int_0^1 \frac{1}{\left(\sqrt{1+x}-\sqrt{x}\right)} \times \frac{\left(\sqrt{1+x}+\sqrt{x}\right)}{\left(\sqrt{1+x}+\sqrt{x}\right)} dx$$

$$= \int_0^1 \frac{\sqrt{1+x}+\sqrt{x}}{1+x-x} dx$$

$$= \int_0^1 \sqrt{1+x}\, dx + \int_0^1 \sqrt{x}\, dx$$

$$= \left[\frac{2}{3}(1+x)^{\frac{3}{2}}\right]_0^1 + \left[\frac{2}{3}(x)^{\frac{3}{2}}\right]_0^1$$

$$= \frac{2}{3}\left[(2)^{\frac{3}{2}} - 1\right] + \frac{2}{3}[1]$$

$$= \frac{2}{3}(2)^{\frac{3}{2}}$$

$$= \frac{2 \cdot 2\sqrt{2}}{3}$$

$$= \frac{4\sqrt{2}}{3}$$

Question 30:

$$\int_0^{\frac{\pi}{4}} \frac{\sin x+\cos x}{9+16\sin 2x}dx$$

Answer

$$\text{Let } I=\int_0^{\frac{\pi}{4}} \frac{\sin x+\cos x}{9+16\sin 2x}dx$$

Also, let $\sin x-\cos x=t \Rightarrow (\cos x+\sin x)dx=dt$

When $x=0,\ t=-1$ and when $x=\frac{\pi}{4},\ t=0$

$$\Rightarrow (\sin x-\cos x)^2=t^2$$

$$\Rightarrow \sin^2 x+\cos^2 x-2\sin x\cos x=t^2$$

$$\Rightarrow 1-\sin 2x=t^2$$

$$\Rightarrow \sin 2x=1-t^2$$

$$\therefore I=\int_{-1}^{0}\frac{dt}{9+16(1-t^2)}$$

$$=\int_{-1}^{0}\frac{dt}{9+16-16t^2}$$

$$=\int_{-1}^{0}\frac{dt}{25-16t^2}=\int_{-1}^{0}\frac{dt}{(5)^2-(4t)^2}$$

$$=\frac{1}{4}\left[\frac{1}{2(5)}\log\left|\frac{5+4t}{5-4t}\right|\right]_{-1}^{0}$$

$$=\frac{1}{40}\left[\log(1)-\log\left|\frac{1}{9}\right|\right]$$

$$=\frac{1}{40}\log 9$$

Question 31:

$$\int_0^{\frac{\pi}{2}} \sin 2x\tan^{-1}(\sin x)\,dx$$

Answer

Let $I=\int_0^{\frac{\pi}{2}}\sin 2x\tan^{-1}(\sin x)dx=\int_0^{\frac{\pi}{2}}2\sin x\cos x\tan^{-1}(\sin x)dx$

Also, let $\sin x=t \Rightarrow \cos x\,dx=dt$

When $x=0, t=0$ and when $x=\frac{\pi}{2}, t=1$

$$\Rightarrow I=2\int_0^1 t\tan^{-1}(t)dt \qquad \ldots(1)$$

Consider $\int t\cdot\tan^{-1}t\,dt=\tan^{-1}t\cdot\int t\,dt-\int\left\{\frac{d}{dt}(\tan^{-1}t)\int t\,dt\right\}dt$

$$=\tan^{-1}t\cdot\frac{t^2}{2}-\int\frac{1}{1+t^2}\cdot\frac{t^2}{2}dt$$

$$=\frac{t^2\tan^{-1}t}{2}-\frac{1}{2}\int\frac{t^2+1-1}{1+t^2}dt$$

$$=\frac{t^2\tan^{-1}t}{2}-\frac{1}{2}\int 1\,dt+\frac{1}{2}\int\frac{1}{1+t^2}dt$$

$$=\frac{t^2\tan^{-1}t}{2}-\frac{1}{2}\cdot t+\frac{1}{2}\tan^{-1}t$$

$$\Rightarrow\int_0^1 t\cdot\tan^{-1}t\,dt=\left[\frac{t^2\cdot\tan^{-1}t}{2}-\frac{t}{2}+\frac{1}{2}\tan^{-1}t\right]_0^1$$

$$=\frac{1}{2}\left[\frac{\pi}{4}-1+\frac{\pi}{4}\right]$$

$$=\frac{1}{2}\left[\frac{\pi}{2}-1\right]=\frac{\pi}{4}-\frac{1}{2}$$

From equation (1), we obtain

$$I=2\left[\frac{\pi}{4}-\frac{1}{2}\right]=\frac{\pi}{2}-1$$

Question 32:

$$\int_0^{\pi}\frac{x\tan x}{\sec x+\tan x}dx$$

Answer

$$Let\ I = \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx \qquad ...(1)$$

$$I = \int_0^{\pi} \left\{ \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)} \right\} dx \qquad \left(\int_0^a f(x)dx = \int_0^a f(a-x)dx \right)$$

$$\Rightarrow I = \int_0^{\pi} \left\{ \frac{-(\pi - x)\tan x}{-(\sec x + \tan x)} \right\} dx$$

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)\tan x}{\sec x + \tan x} dx \qquad ...(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\pi} \frac{\pi \tan x}{\sec x + \tan x} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x} + \frac{\sin x}{\cos x}} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{\sin x + 1 - 1}{1 + \sin x} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} 1.dx - \pi \int_0^{\pi} \frac{1}{1 + \sin x} dx$$

$$\Rightarrow 2I = \pi [x]_0^{\pi} - \pi \int_0^{\pi} \frac{1 - \sin x}{\cos^2 x} dx$$

$$\Rightarrow 2I = \pi^2 - \pi \int_0^{\pi} (\sec^2 x - \tan x \sec x) dx$$

$$\Rightarrow 2I = \pi^2 - \pi [\tan x - \sec x]_0^{\pi}$$

$$\Rightarrow 2I = \pi^2 - \pi [\tan \pi - \sec \pi - \tan 0 + \sec 0]$$

$$\Rightarrow 2I = \pi^2 - \pi [0 - (-1) - 0 + 1]$$

$$\Rightarrow 2I = \pi^2 - 2\pi$$

$$\Rightarrow 2I = \pi(\pi - 2)$$

$$\Rightarrow I = \frac{\pi}{2}(\pi - 2)$$

Question 33:

$$\int_1^4 \left[|x-1|+|x-2|+|x-3|\right]dx$$

Answer

Let $I = \int_1^4 \left[|x-1|+|x-2|+|x-3|\right]dx$

$\Rightarrow I = \int_1^4 |x-1|\,dx + \int_1^4 |x-2|\,dx + \int_1^4 |x-3|\,dx$

$I = I_1 + I_2 + I_3$ ...(1)

where, $I_1 = \int_1^4 |x-1|\,dx$, $I_2 = \int_1^4 |x-2|\,dx$, and $I_3 = \int_1^4 |x-3|\,dx$

$I_1 = \int_1^4 |x-1|\,dx$

$(x-1) \geq 0$ for $1 \leq x \leq 4$

$\therefore I_1 = \int_1^4 (x-1)\,dx$

$\Rightarrow I_1 = \left[\frac{x^2}{x} - x\right]_1^4$

$\Rightarrow I_1 = \left[8-4-\frac{1}{2}+1\right] = \frac{9}{2}$ ...(2)

$I_2 = \int_1^4 |x-2|\,dx$

$x-2 \geq 0$ for $2 \leq x \leq 4$ and $x-2 \leq 0$ for $1 \leq x \leq 2$

$\therefore I_2 = \int_1^2 (2-x)\,dx + \int_2^4 (x-2)\,dx$

$\Rightarrow I_2 = \left[2x - \frac{x^2}{2}\right]_1^2 + \left[\frac{x^2}{2} - 2x\right]_2^4$

$\Rightarrow I_2 = \left[4-2-2+\frac{1}{2}\right] + \left[8-8-2+4\right]$

$\Rightarrow I_2 = \frac{1}{2} + 2 = \frac{5}{2}$ ...(3)

$$I_3 = \int_1^4 |x-3|\,dx$$

$x-3 \ge 0$ for $3 \le x \le 4$ and $x-3 \le 0$ for $1 \le x \le 3$

$$\therefore I_3 = \int_1^3 (3-x)\,dx + \int_3^4 (x-3)\,dx$$

$$\Rightarrow I_3 = \left[3x - \frac{x^2}{2}\right]_1^3 + \left[\frac{x^2}{2} - 3x\right]_3^4$$

$$\Rightarrow I_3 = \left[9 - \frac{9}{2} - 3 + \frac{1}{2}\right] + \left[8 - 12 - \frac{9}{2} + 9\right]$$

$$\Rightarrow I_3 = [6-4] + \left[\frac{1}{2}\right] = \frac{5}{2} \qquad \ldots(4)$$

From equations (1), (2), (3), and (4), we obtain

$$I = \frac{9}{2} + \frac{5}{2} + \frac{5}{2} = \frac{19}{2}$$

Question 34:

$$\int_1^3 \frac{dx}{x^2(x+1)} = \frac{2}{3} + \log\frac{2}{3}$$

Answer

$$\text{Let } I = \int_1^3 \frac{dx}{x^2(x+1)}$$

$$\text{Also, let } \frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$\Rightarrow 1 = Ax(x+1) + B(x+1) + C(x^2)$$

$$\Rightarrow 1 = Ax^2 + Ax + Bx + B + Cx^2$$

Equating the coefficients of x^2, x, and constant term, we obtain

$A + C = 0$

$A + B = 0$

$B = 1$

On solving these equations, we obtain

$A = -1$, $C = 1$, and $B = 1$

$$\therefore \frac{1}{x^2(x+1)} = \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{(x+1)}$$

$$\Rightarrow I = \int_1^3 \left\{ -\frac{1}{x} + \frac{1}{x^2} + \frac{1}{(x+1)} \right\} dx$$

$$= \left[-\log x - \frac{1}{x} + \log(x+1) \right]_1^3$$

$$= \left[\log\left(\frac{x+1}{x}\right) - \frac{1}{x} \right]_1^3$$

$$= \log\left(\frac{4}{3}\right) - \frac{1}{3} - \log\left(\frac{2}{1}\right) + 1$$

$$= \log 4 - \log 3 - \log 2 + \frac{2}{3}$$

$$= \log 2 - \log 3 + \frac{2}{3}$$

$$= \log\left(\frac{2}{3}\right) + \frac{2}{3}$$

Hence, the given result is proved.

Question 35:

$$\int_0^1 xe^x dx = 1$$

Answer

Let $I = \int_0^1 xe^x \, dx$

Integrating by parts, we obtain

$$I = x\int_0^1 e^x dx - \int_0^1 \left\{ \left(\frac{d}{dx}(x) \right) \int e^x dx \right\} dx$$

$$= \left[xe^x \right]_0^1 - \int_0^1 e^x dx$$

$$= \left[xe^x \right]_0^1 - \left[e^x \right]_0^1$$

$$= e - e + 1$$

$$= 1$$

Hence, the given result is proved.

Question 36:

$$\int_{-1}^{1} x^{17}\cos^4 x dx = 0$$

Answer

$$\text{Let } I = \int_{-1}^{1} x^{17}\cos^4 x dx$$

$$\text{Also, let } f(x) = x^{17}\cos^4 x$$

$$\Rightarrow f(-x) = (-x)^{17}\cos^4(-x) = -x^{17}\cos^4 x = -f(x)$$

Therefore, $f(x)$ is an odd function.

It is known that if $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)dx = 0$

$$\therefore I = \int_{-1}^{1} x^{17}\cos^4 x\, dx = 0$$

Hence, the given result is proved.

Question 37:

$$\int_0^{\frac{\pi}{2}} \sin^3 x\, dx = \frac{2}{3}$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \sin^3 x\, dx$$

$$I = \int_0^{\frac{\pi}{2}} \sin^2 x \cdot \sin x\, dx$$

$$= \int_0^{\frac{\pi}{2}} \left(1-\cos^2 x\right)\sin x\, dx$$

$$= \int_0^{\frac{\pi}{2}} \sin x\, dx - \int_0^{\frac{\pi}{2}} \cos^2 x \cdot \sin x\, dx$$

$$= \left[-\cos x\right]_0^{\frac{\pi}{2}} + \left[\frac{\cos^3 x}{3}\right]_0^{\frac{\pi}{2}}$$

$$= 1 + \frac{1}{3}\left[-1\right] = 1 - \frac{1}{3} = \frac{2}{3}$$

Hence, the given result is proved.

Question 38:

$$\int_0^{\frac{\pi}{4}} 2\tan^3 x dx = 1 - \log 2$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{4}} 2\tan^3 x\, dx$$

$$I = 2\int_0^{\frac{\pi}{4}} \tan^2 x \tan x\, dx = 2\int_0^{\frac{\pi}{4}} \left(\sec^2 x - 1\right)\tan x\, dx$$

$$= 2\int_0^{\frac{\pi}{4}} \sec^2 x \tan x\, dx - 2\int_0^{\frac{\pi}{4}} \tan x\, dx$$

$$= 2\left[\frac{\tan^2 x}{2}\right]_0^{\frac{\pi}{4}} + 2\left[\log\cos x\right]_0^{\frac{\pi}{4}}$$

$$= 1 + 2\left[\log\cos\frac{\pi}{4} - \log\cos 0\right]$$

$$= 1 + 2\left[\log\frac{1}{\sqrt{2}} - \log 1\right]$$

$$= 1 - \log 2 - \log 1 = 1 - \log 2$$

Hence, the given result is proved.

Question 39:

$$\int_0^1 \sin^{-1} x\, dx = \frac{\pi}{2} - 1$$

Answer

$$\text{Let } I = \int_0^1 \sin^{-1} x\, dx$$

$$\Rightarrow I = \int_0^1 \sin^{-1} x \cdot 1 \cdot dx$$

Integrating by parts, we obtain

$$I=\left[\sin^{-1}x\cdot x\right]_0^1-\int_0^1\frac{1}{\sqrt{1-x^2}}\cdot x\,dx$$

$$=\left[x\sin^{-1}x\right]_0^1+\frac{1}{2}\int_0^1\frac{(-2x)}{\sqrt{1-x^2}}dx$$

Let $1 - x^2 = t$ □ $-2x\,dx = dt$

When $x=0,\ t=1$ and when $x=1,\ t=0$

$$I=\left[x\sin^{-1}x\right]_0^1+\frac{1}{2}\int_1^0\frac{dt}{\sqrt{t}}$$

$$=\left[x\sin^{-1}x\right]_0^1+\frac{1}{2}\left[2\sqrt{t}\right]_1^0$$

$$=\sin^{-1}(1)+\left[-\sqrt{1}\right]$$

$$=\frac{\pi}{2}-1$$

Hence, the given result is proved.

Question 40:

Evaluate $\int_0^1 e^{2-3x}dx$ as a limit of a sum.

Answer

Let $I=\int_0^1 e^{2-3x}dx$

It is known that,

$$\int_a^b f(x)dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+\ldots+f(a+(n-1)h)\right]$$

Where, $h=\dfrac{b-a}{n}$

Here, $a=0, b=1,$ and $f(x)=e^{2-3x}$

$$\Rightarrow h=\frac{1-0}{n}=\frac{1}{n}$$

$$\therefore \int_0^1 e^{2-3x}dx=(1-0)\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f(0+h)+\ldots+f(0+(n-1)h)\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[e^2+e^{2-3h}+\ldots e^{2-3(n-1)h}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[e^2\left\{1+e^{-3h}+e^{-6h}+e^{-9h}+\ldots e^{-3(n-1)h}\right\}\right]$$

$$= \lim_{h\to\infty} \frac{1}{n}\left[e^2\left\{\frac{1-\left(e^{-3h}\right)^n}{1-\left(e^{-3h}\right)}\right\}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[e^2\left\{\frac{1-e^{-\frac{3}{n}\times n}}{1-e^{-\frac{3}{n}}}\right\}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{e^2\left(1-e^{-3}\right)}{1-e^{-\frac{3}{n}}}\right]$$

$$= e^2\left(e^{-3}-1\right)\lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{e^{-\frac{3}{n}}-1}\right]$$

$$= e^2\left(e^{-3}-1\right)\lim_{n\to\infty}\left(-\frac{1}{3}\right)\left[\frac{-\frac{3}{n}}{e^{-\frac{3}{n}}-1}\right]$$

$$= \frac{-e^2\left(e^{-3}-1\right)}{3}\lim_{n\to\infty}\left[\frac{-\frac{3}{n}}{e^{-\frac{3}{n}}-1}\right]$$

$$= \frac{-e^2\left(e^{-3}-1\right)}{3}(1) \qquad \left[\lim_{n\to\infty}\frac{x}{e^x-1}\right]$$

$$= \frac{-e^{-1}+e^2}{3}$$

$$= \frac{1}{3}\left(e^2-\frac{1}{e}\right)$$

Question 41:

$\int \frac{dx}{e^x+e^{-x}}$ is equal to

A. $\tan^{-1}\left(e^x\right)+C$

B. $\tan^{-1}\left(e^{-x}\right)+C$

C. $\log\left(e^{x}-e^{-x}\right)+C$

D. $\log\left(e^{x}+e^{-x}\right)+C$

Answer

$$\text{Let } I=\int\frac{dx}{e^{x}+e^{-x}}dx=\int\frac{e^{x}}{e^{2x}+1}dx$$

$$\text{Also, let } e^{x}=t \Rightarrow e^{x}\,dx=dt$$

$$\therefore I=\int\frac{dt}{1+t^{2}}$$

$$=\tan^{-1}t+C$$

$$=\tan^{-1}\left(e^{x}\right)+C$$

Hence, the correct Answer is A.

Question 42:

$\int\frac{\cos 2x}{(\sin x+\cos x)^{2}}dx$ is equal to

A. $\frac{-1}{\sin x+\cos x}+C$

B. $\log|\sin x+\cos x|+C$

C. $\log|\sin x-\cos x|+C$

D. $\frac{1}{(\sin x+\cos x)^{2}}$

Answer

$$\text{Let } I = \frac{\cos 2x}{(\cos x + \sin x)^2}$$

$$I = \int \frac{\cos^2 x - \sin^2 x}{(\cos x + \sin x)^2} dx$$

$$= \int \frac{(\cos x + \sin x)(\cos x - \sin x)}{(\cos x + \sin x)^2} dx$$

$$= \int \frac{\cos x - \sin x}{\cos + \sin x} dx$$

$$\text{Let } \cos x + \sin x = t \Rightarrow (\cos x - \sin x) dx = dt$$

$$\therefore I = \int \frac{dt}{t}$$

$$= \log|t| + C$$

$$= \log|\cos x + \sin x| + C$$

Hence, the correct Answer is B.

Question 43:

If $f(a+b-x) = f(x)$, then $\int_a^b x f(x) dx$ is equal to

A. $\frac{a+b}{2} \int_a^b f(b-x) dx$

B. $\frac{a+b}{2} \int_a^b f(b+x) dx$

C. $\frac{b-a}{2} \int_a^b f(x) dx$

D. $\frac{a+b}{2} \int_a^b f(x) dx$

Answer

$$\text{Let } I = \int_a^b x f(x) dx \qquad ...(1)$$

$$I=\int_a^b (a+b-x)f(a+b-x)\,dx \qquad \left(\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx\right)$$

$$\Rightarrow I=\int_a^b (a+b-x)f(x)\,dx$$

$$\Rightarrow I=(a+b)\int_a^b f(x)\,dx \quad -I \qquad [\text{Using}(1)]$$

$$\Rightarrow I+I=(a+b)\int_a^b f(x)\,dx$$

$$\Rightarrow 2I=(a+b)\int_a^b f(x)\,dx$$

$$\Rightarrow I=\left(\frac{a+b}{2}\right)\int_a^b f(x)\,dx$$

Hence, the correct Answer is D.

Question 44:

The value of $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$ is

A. 1

B. 0

C. − 1

D. $\frac{\pi}{4}$

Answer

$$\text{Let } I=\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$$

$$\Rightarrow I=\int_0^1 \tan^{-1}\left(\frac{x-(1-x)}{1+x(1-x)}\right)dx$$

$$\Rightarrow I=\int_0^1\left[\tan^{-1}x-\tan^{-1}(1-x)\right]dx \qquad ...(1)$$

$$\Rightarrow I=\int_0^1\left[\tan^{-1}(1-x)-\tan^{-1}(1-1+x)\right]dx$$

$$\Rightarrow I=\int_0^1\left[\tan^{-1}(1-x)-\tan^{-1}(x)\right]dx$$

$$\Rightarrow I=\int_0^1\left[\tan^{-1}(1-x)-\tan^{-1}(x)\right]dx \qquad ...(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^1 \left(\tan^{-1} x + \tan^{-1}(1-x) - \tan^{-1}(1-x) - \tan^{-1} x\right) dx$$

$$\Rightarrow 2I = 0$$

$$\Rightarrow I = 0$$

Hence, the correct Answer is B.