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Question 1:

Using differentials, find the approximate value of each of the following.

(a) $\left(\frac{17}{81}\right)^{\frac{1}{4}}$ (b) $(33)^{-\frac{1}{5}}$

Answer

(a) Consider $y = x^{\frac{1}{4}}$. Let $x = \frac{16}{81}$ and $\Delta x = \frac{1}{81}$.

Then, $\Delta y = (x + \Delta x)^{\frac{1}{4}} - x^{\frac{1}{4}}$

$$= \left(\frac{17}{81}\right)^{\frac{1}{4}} - \left(\frac{16}{81}\right)^{\frac{1}{4}}$$

$$= \left(\frac{17}{81}\right)^{\frac{1}{4}} - \frac{2}{3}$$

$$\therefore \left(\frac{17}{81}\right)^{\frac{1}{4}} = \frac{2}{3} + \Delta y$$

Now, *dy* is approximately equal to Δ*y* and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{4(x)^{\frac{3}{4}}}(\Delta x) \qquad \left(\text{as } y = x^{\frac{1}{4}}\right)$$

$$= \frac{1}{4\left(\frac{16}{81}\right)^{\frac{3}{4}}}\left(\frac{1}{81}\right) = \frac{27}{4\times 8}\times\frac{1}{81} = \frac{1}{32\times 3} = \frac{1}{96} = 0.010$$

Hence, the approximate value of $\left(\frac{17}{81}\right)^{\frac{1}{4}}$ is $\frac{2}{3} + 0.010$ = 0.667 + 0.010 = 0.677.

(b) Consider $y = x^{-\frac{1}{5}}$. Let x = 32 and Δx = 1.

Then, $\Delta y=(x+\Delta x)^{-\frac{1}{5}}-x^{-\frac{1}{5}}=(33)^{-\frac{1}{5}}-(32)^{-\frac{1}{5}}=(33)^{-\frac{1}{5}}-\frac{1}{2}$

$$\therefore (33)^{-\frac{1}{5}}=\frac{1}{2}+\Delta y$$

Now, *dy* is approximately equal to Δ*y* and is given by,

$$dy=\left(\frac{dy}{dx}\right)(\Delta x)=\frac{-1}{5(x)^{\frac{6}{5}}}(\Delta x) \qquad \left(\text{as } y=x^{-\frac{1}{5}}\right)$$

$$=-\frac{1}{5(2)^6}(1)=-\frac{1}{320}=-0.003$$

Hence, the approximate value of $(33)^{-\frac{1}{5}}$ is $\frac{1}{2}+(-0.003)$

= 0.5 − 0.003 = 0.497.

Question 2:

Show that the function given by $f(x)=\frac{\log x}{x}$ has maximum at *x* = *e*.

Answer

The given function is $f(x)=\frac{\log x}{x}$.

$$f'(x)=\frac{x\left(\frac{1}{x}\right)-\log x}{x^2}=\frac{1-\log x}{x^2}$$

Now, $f'(x)=0$

⇒ 1 − log *x* = 0

$\Rightarrow \log x = 1$

$\Rightarrow \log x = \log e$

$\Rightarrow x = e$

$$\text{Now, } f''(x) = \frac{x^2\left(-\frac{1}{x}\right) - (1-\log x)(2x)}{x^4}$$

$$= \frac{-x - 2x(1-\log x)}{x^4}$$

$$= \frac{-3 + 2\log x}{x^3}$$

$$\text{Now, } f''(e) = \frac{-3 + 2\log e}{e^3} = \frac{-3+2}{e^3} = \frac{-1}{e^3} < 0$$

Therefore, by second derivative test, f is the maximum at $x = e$.

Question 3:

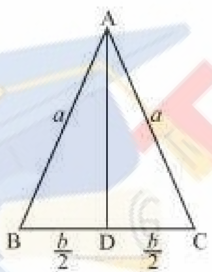
The two equal sides of an isosceles triangle with fixed base *b* are decreasing at the rate of 3 cm per second. How fast is the area decreasing when the two equal sides are equal to the base?

Answer

Let ΔABC be isosceles where BC is the base of fixed length *b*.

Let the length of the two equal sides of ΔABC be *a*.

Draw AD□BC.



Now, in ΔADC, by applying the Pythagoras theorem, we have:

$$\text{AD} = \sqrt{a^2 - \frac{b^2}{4}}$$

□ Area of triangle $(A)=\frac{1}{2}b\sqrt{a^2-\frac{b^2}{4}}$

The rate of change of the area with respect to time (t) is given by,

$$\frac{dA}{dt}=\frac{1}{2}b\cdot\frac{2a}{2\sqrt{a^2-\frac{b^2}{4}}}\frac{da}{dt}=\frac{ab}{\sqrt{4a^2-b^2}}\frac{da}{dt}$$

It is given that the two equal sides of the triangle are decreasing at the rate of 3 cm per second.

□ $\frac{da}{dt}=-3\text{ cm/s}$

$$\therefore\frac{dA}{dt}=\frac{-3ab}{\sqrt{4a^2-b^2}}$$

Then, when $a = b$, we have:

$$\frac{dA}{dt}=\frac{-3b^2}{\sqrt{4b^2-b^2}}=\frac{-3b^2}{\sqrt{3b^2}}=-\sqrt{3}b$$

Hence, if the two equal sides are equal to the base, then the area of the triangle is decreasing at the rate of $\sqrt{3}\,b\text{ cm}^2/\text{s}$.

Question 4:

Find the equation of the normal to curve $y^2 = 4x$ at the point (1, 2).

Answer

The equation of the given curve is $y^2=4x$.

Differentiating with respect to x, we have:

$$2y\frac{dy}{dx}=4$$

$$\Rightarrow\frac{dy}{dx}=\frac{4}{2y}=\frac{2}{y}$$

$$\therefore\left.\frac{dy}{dx}\right]_{(1,2)}=\frac{2}{2}=1$$

Now, the slope of the normal at point (1, 2) is $\dfrac{-1}{\left.\dfrac{dy}{dx}\right]_{(1,2)}} = \dfrac{-1}{1} = -1.$

□Equation of the normal at (1, 2) is $y - 2 = -1(x - 1)$.

□ $y - 2 = -x + 1$

□ $x + y - 3 = 0$

Question 5:

Show that the normal at any point θ to the curve

$x = a\cos\theta + a\theta\sin\theta,\ y = a\sin\theta - a\theta\cos\theta$ is at a constant distance from the origin.

Answer

We have $x = a\cos\theta + a\theta\sin\theta$.

$$\therefore \frac{dx}{d\theta} = -a\sin\theta + a\sin\theta + a\theta\cos\theta = a\theta\cos\theta$$

$$y = a\sin\theta - a\theta\cos\theta$$

$$\therefore \frac{dy}{d\theta} = a\cos\theta - a\cos\theta + a\theta\sin\theta = a\theta\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

□ Slope of the normal at any point θ is $-\dfrac{1}{\tan\theta}$.

The equation of the normal at a given point (x, y) is given by,

$$y - a\sin\theta + a\theta\cos\theta = \frac{-1}{\tan\theta}\left(x - a\cos\theta - a\theta\sin\theta\right)$$

$$\Rightarrow y\sin\theta - a\sin^2\theta + a\theta\sin\theta\cos\theta = -x\cos\theta + a\cos^2\theta + a\theta\sin\theta\cos\theta$$

$$\Rightarrow x\cos\theta + y\sin\theta - a\left(\sin^2\theta + \cos^2\theta\right) = 0$$

$$\Rightarrow x\cos\theta + y\sin\theta - a = 0$$

Now, the perpendicular distance of the normal from the origin is

$\dfrac{|-a|}{\sqrt{\cos^2\theta + \sin^2\theta}} = \dfrac{|-a|}{\sqrt{1}} = |-a|$, which is independent of θ.

Hence, the perpendicular distance of the normal from the origin is constant.

Question 6:

Find the intervals in which the function *f* given by

$$f(x)=\frac{4\sin x-2x-x\cos x}{2+\cos x}$$

is (i) increasing (ii) decreasing

Answer

$$f(x)=\frac{4\sin x-2x-x\cos x}{2+\cos x}$$

$$\therefore f'(x)=\frac{(2+\cos x)(4\cos x-2-\cos x+x\sin x)-(4\sin x-2x-x\cos x)(-\sin x)}{(2+\cos x)^2}$$

$$=\frac{(2+\cos x)(3\cos x-2+x\sin x)+\sin x(4\sin x-2x-x\cos x)}{(2+\cos x)^2}$$

$$=\frac{6\cos x-4+2x\sin x+3\cos^2 x-2\cos x+x\sin x\cos x+4\sin^2 x-2x\sin x-x\sin x\cos x}{(2+\cos x)^2}$$

$$=\frac{4\cos x-4+3\cos^2 x+4\sin^2 x}{(2+\cos x)^2}$$

$$=\frac{4\cos x-4+3\cos^2 x+4-4\cos^2 x}{(2+\cos x)^2}$$

$$=\frac{4\cos x-\cos^2 x}{(2+\cos x)^2}=\frac{\cos x(4-\cos x)}{(2+\cos x)^2}$$

Now, $f'(x)=0$

$\Rightarrow$ cos *x* = 0 or cos *x* = 4

But, cos *x* ≠ 4

□cos *x* = 0

$\Rightarrow x=\frac{\pi}{2},\frac{3\pi}{2}$

Now, $x=\frac{\pi}{2}$ and $x=\frac{3\pi}{2}$ divides (0, 2π) into three disjoint intervals i.e.,

$\left(0,\frac{\pi}{2}\right),\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$, and $\left(\frac{3\pi}{2},2\pi\right)$.

In intervals $\left(0,\frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2},2\pi\right)$, $f'(x)>0$.

Thus, $f(x)$ is increasing for $0<x<\frac{x}{2}$ and $\frac{3\pi}{2}<x<2\pi$.

In the interval $\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$, $f'(x)<0$.

Thus, $f(x)$ is decreasing for $\frac{\pi}{2}<x<\frac{3\pi}{2}$.

Question 7:

Find the intervals in which the function f given by $f(x)=x^3+\frac{1}{x^3}, x\neq 0$ is

(i) increasing (ii) decreasing

Answer

$$f(x)=x^3+\frac{1}{x^3}$$

$$\therefore f'(x)=3x^2-\frac{3}{x^4}=\frac{3x^6-3}{x^4}$$

Then, $f'(x)=0\Rightarrow 3x^6-3=0\Rightarrow x^6=1\Rightarrow x=\pm 1$

Now, the points $x = 1$ and $x = -1$ divide the real line into three disjoint intervals

i.e., $(-\infty,-1),(-1,1)$, and $(1,\infty)$.

In intervals $(-\infty,-1)$ and $(1,\infty)$ i.e., when $x < -1$ and $x > 1$, $f'(x)>0$.

Thus, when $x < -1$ and $x > 1$, f is increasing.

In interval $(-1, 1)$ i.e., when $-1 < x < 1$, $f'(x)<0$.

Thus, when $-1 < x < 1$, f is decreasing.

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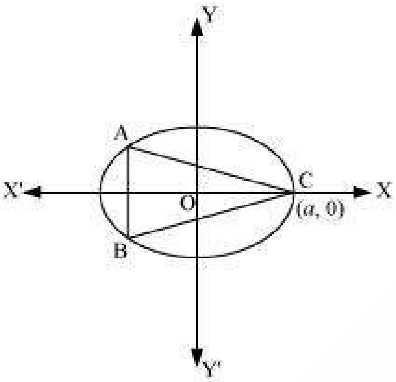
Question 8:

Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with its vertex at one end of the major axis.

Answer



The given ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let the major axis be along the x −axis.

Let ABC be the triangle inscribed in the ellipse where vertex C is at (a, 0).

Since the ellipse is symmetrical with respect to the x−axis and y −axis, we can assume the coordinates of A to be $(-x_1, y_1)$ and the coordinates of B to be $(-x_1, -y_1)$.

Now, we have $y_1=\pm\frac{b}{a}\sqrt{a^2-x_1^2}$.

☐Coordinates of A are $\left(-x_1, \frac{b}{a}\sqrt{a^2-x_1^2}\right)$ and the coordinates of B are $\left(x_1, -\frac{b}{a}\sqrt{a^2-x_1^2}\right)$.

As the point (x_1, y_1) lies on the ellipse, the area of triangle ABC (A) is given by,

$$A=\frac{1}{2}\left|a\left(\frac{2b}{a}\sqrt{a^2-x_1^2}\right)+(-x_1)\left(-\frac{b}{a}\sqrt{a^2-x_1^2}\right)+(-x_1)\left(-\frac{b}{a}\sqrt{a^2-x_1^2}\right)\right|$$

$$\Rightarrow A=b\sqrt{a^2-x_1^2}+x_1\frac{b}{a}\sqrt{a^2-x_1^2} \qquad ...(1)$$

$$\therefore \frac{dA}{dx_1}=\frac{-2x_1b}{2\sqrt{a^2-x_1^2}}+\frac{b}{a}\sqrt{a^2-x_1^2}-\frac{2bx_1^2}{a2\sqrt{a^2-x_1^2}}$$

$$=\frac{b}{a\sqrt{a^2-x_1^2}}\left[-x_1a+\left(a^2-x_1^2\right)-x_1^2\right]$$

$$=\frac{b\left(-2x_1^2-x_1a+a^2\right)}{a\sqrt{a^2-x_1^2}}$$

Now, $\frac{dA}{dx_1}=0$

$$\Rightarrow -2x_1^2-x_1a+a^2=0$$

$$\Rightarrow x_1=\frac{a\pm\sqrt{a^2-4(-2)\left(a^2\right)}}{2(-2)}$$

$$=\frac{a\pm\sqrt{9a^2}}{-4}$$

$$=\frac{a\pm 3a}{-4}$$

$$\Rightarrow x_1=-a,\ \frac{a}{2}$$

But, x_1 cannot be equal to a.

$$\therefore x_1 = \frac{a}{2} \Rightarrow y_1 = \frac{b}{a}\sqrt{a^2 - \frac{a^2}{4}} = \frac{ba}{2a}\sqrt{3} = \frac{\sqrt{3}b}{2}$$

Now, $$\frac{d^2A}{dx_1^2} = \frac{b}{a}\left\{\frac{\sqrt{a^2-x_1^2}\,(-4x_1-a)-\left(-2x_1^2-x_1a+a^2\right)\frac{(-2x_1)}{2\sqrt{a^2-x_1^2}}}{a^2-x_1^2}\right\}$$

$$= \frac{b}{a}\left\{\frac{\left(a^2-x_1^2\right)(-4x_1-a)+x_1\left(-2x_1^2-x_1a+a^2\right)}{\left(a^2-x_1^2\right)^{\frac{3}{2}}}\right\}$$

$$= \frac{b}{a}\left\{\frac{2x^3-3a^2x-a^3}{\left(a^2-x_1^2\right)^{\frac{3}{2}}}\right\}$$

Also, when $x_1 = \frac{a}{2}$, then

$$\frac{d^2A}{dx_1^2} = \frac{b}{a}\left\{\frac{2\frac{a^3}{8}-3\frac{a^3}{2}-a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}}\right\} = \frac{b}{a}\left\{\frac{\frac{a^3}{4}-\frac{3}{2}a^3-a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}}\right\}$$

$$= -\frac{b}{a}\left\{\frac{\frac{9}{4}a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}}\right\} < 0$$

Thus, the area is the maximum when $x_1 = \frac{a}{2}$.

☐ Maximum area of the triangle is given by,

$$A = b\sqrt{a^2 - \frac{a^2}{4}} + \left(\frac{a}{2}\right)\frac{b}{a}\sqrt{a^2 - \frac{a^2}{4}}$$

$$= ab\frac{\sqrt{3}}{2} + \left(\frac{a}{2}\right)\frac{b}{a} \times \frac{a\sqrt{3}}{2}$$

$$= \frac{ab\sqrt{3}}{2} + \frac{ab\sqrt{3}}{4} = \frac{3\sqrt{3}}{4}ab$$

Question 9:

A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3. If building of tank costs Rs 70 per sq meters for the base and Rs 45 per square metre for sides. What is the cost of least expensive tank?

Answer

Let *l*, *b*, and *h* represent the length, breadth, and height of the tank respectively.

Then, we have height (*h)* = 2 m

Volume of the tank = 8m^3

Volume of the tank = *l* × *b* × *h*

☐ 8 = *l* × *b* × 2

$$\Rightarrow lb = 4 \Rightarrow b = \frac{4}{l}$$

Now, area of the base = *lb* = 4

Area of the 4 walls (*A)* = 2*h* (*l* + *b*)

$$\therefore A = 4\left(l + \frac{4}{l}\right)$$

$$\Rightarrow \frac{dA}{dl} = 4\left(1 - \frac{4}{l^2}\right)$$

$$\text{Now, } \frac{dA}{dl} = 0$$

$$\Rightarrow 1 - \frac{4}{l^2} = 0$$

$$\Rightarrow l^2 = 4$$

$$\Rightarrow l = \pm 2$$

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However, the length cannot be negative.

Therefore, we have l = 4.

$$\therefore b = \frac{4}{l} = \frac{4}{2} = 2$$

$$\text{Now, } \frac{d^2A}{dl^2} = \frac{32}{l^3}$$

$$\text{When } l = 2,\ \frac{d^2A}{dl^2} = \frac{32}{8} = 4 > 0.$$

Thus, by second derivative test, the area is the minimum when l = 2.

We have l = b = h = 2.

☐Cost of building the base = Rs 70 × (lb) = Rs 70 (4) = Rs 280

Cost of building the walls = Rs 2h (l + b) × 45 = Rs 90 (2) (2 + 2)

= Rs 8 (90) = Rs 720

Required total cost = Rs (280 + 720) = Rs 1000

Hence, the total cost of the tank will be Rs 1000.

Question 10:

The sum of the perimeter of a circle and square is k, where k is some constant. Prove that the sum of their areas is least when the side of square is double the radius of the circle.

Answer

Let r be the radius of the circle and a be the side of the square.

Then, we have:

$$2\pi r + 4a = k \text{ (where } k \text{ is constant)}$$

$$\Rightarrow a = \frac{k - 2\pi r}{4}$$

The sum of the areas of the circle and the square (A) is given by,

$$A = \pi r^2 + a^2 = \pi r^2 + \frac{(k - 2\pi r)^2}{16}$$

$$\therefore \frac{dA}{dr} = 2\pi r + \frac{2(k - 2\pi r)(-2\pi)}{16} = 2\pi r - \frac{\pi(k - 2\pi r)}{4}$$

$$\text{Now}, \frac{dA}{dr} = 0$$

$$\Rightarrow 2\pi r = \frac{\pi(k - 2\pi r)}{4}$$

$$8r = k - 2\pi r$$

$$\Rightarrow (8 + 2\pi)r = k$$

$$\Rightarrow r = \frac{k}{8 + 2\pi} = \frac{k}{2(4 + \pi)}$$

$$\text{Now}, \ \frac{d^2A}{dr^2} = 2\pi + \frac{\pi^2}{2} > 0$$

$$\therefore \text{When } r = \frac{k}{2(4\pi +\ )}, \ \frac{d^2A}{dr^2} > 0.$$

☐ The sum of the areas is least when $r = \frac{k}{2(4\pi +\ )}$.

$$\text{When } r = \frac{k}{2(4\pi +\ )}, \ a = \frac{k - 2\pi\left[\frac{k}{2(4\pi +\ )}\right]}{4} = \frac{k(4\pi +\ )\pi \ k}{4 \ 4(\ \pi +\ )} = \frac{4k}{4 \ 4(\ \pi +\ )} = \frac{k}{4 \ \pi +} = 2r.$$

Hence, it has been proved that the sum of their areas is least when the side of the square is double the radius of the circle.

Question 11:

A window is in the form of rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.

Answer

Let x and y be the length and breadth of the rectangular window.

Radius of the semicircular opening $= \frac{x}{2}$

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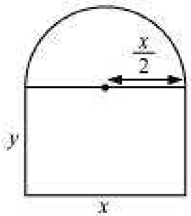


It is given that the perimeter of the window is 10 m.

$$\therefore x + 2y + \frac{\pi x}{2} = 10$$

$$\Rightarrow x\left(1+\frac{\pi}{2}\right) + 2y = 10$$

$$\Rightarrow 2y = 10 - x\left(1+\frac{\pi}{2}\right)$$

$$\Rightarrow y = 5 - x\left(\frac{1}{2}+\frac{\pi}{4}\right)$$

☐Area of the window *(A)* is given by,

$$A = xy + \frac{\pi}{2}\left(\frac{x}{2}\right)^2$$

$$= x\left[5 - x\left(\frac{1}{2}+\frac{\pi}{4}\right)\right] + \frac{\pi}{8}x^2$$

$$= 5x - x^2\left(\frac{1}{2}+\frac{\pi}{4}\right) + \frac{\pi}{8}x^2$$

$$\therefore \frac{dA}{dx} = 5 - 2x\left(\frac{1}{2}+\frac{\pi}{4}\right) + \frac{\pi}{4}x$$

$$= 5 - x\left(1+\frac{\pi}{2}\right) + \frac{\pi}{4}x$$

$$\therefore \frac{d^2A}{dx^2} = -\left(1+\frac{\pi}{2}\right) + \frac{\pi}{4} = -1 - \frac{\pi}{4}$$

Now, $\frac{dA}{dx}=0$

$$\Rightarrow 5-x\left(1+\frac{\pi}{2}\right)+\frac{\pi}{4}x=0$$

$$\Rightarrow 5-x-\frac{\pi}{4}x=0$$

$$\Rightarrow x\left(1+\frac{\pi}{4}\right)=5$$

$$\Rightarrow x=\frac{5}{\left(1+\frac{\pi}{4}\right)}=\frac{20}{\pi+4}$$

Thus, when $x=\frac{20}{\pi+4}$ then $\frac{d^2A}{dx^2}<0.$

Therefore, by second derivative test, the area is the maximum when length $x=\frac{20}{\pi+4}$ m.

Now,

$$y=5-\frac{20}{\pi+4}\left(\frac{2+\pi}{4}\right)=5-\frac{5(2+\pi)}{\pi+4}=\frac{10}{\pi+4}\text{ m}$$

Hence, the required dimensions of the window to admit maximum light is given by $\text{length}=\frac{20}{\pi+4}$ m and breadth $=\frac{10}{\pi+4}$ m.

Question 12:

A point on the hypotenuse of a triangle is at distance *a* and *b* from the sides of the triangle.

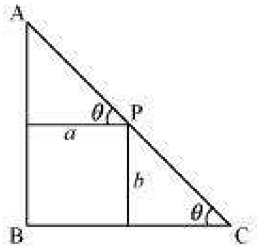
Show that the minimum length of the hypotenuse is $\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{3}{2}}$

Answer

Let ΔABC be right-angled at B. Let AB = *x* and BC = *y*.

Let P be a point on the hypotenuse of the triangle such that P is at a distance of *a* and *b* from the sides AB and BC respectively.

Let □C = θ.



We have,

$$AC = \sqrt{x^2 + y^2}$$

Now,

PC = b cosec θ

And, AP = a sec θ

□AC = AP + PC

□ AC = b cosec θ + a sec θ … (1)

$$\therefore \frac{d(AC)}{d\theta} = -b\operatorname{cosec}\theta\cot\theta + a\sec\theta\tan\theta$$

$$\therefore \frac{d(AC)}{d\theta} = 0$$

$$\Rightarrow a\sec\theta\tan\theta = b\operatorname{cosec}\theta\cot\theta$$

$$\Rightarrow \frac{a}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta} = \frac{b}{\sin\theta}\frac{\cos\theta}{\sin\theta}$$

$$\Rightarrow a\sin^3\theta = b\cos^3\theta$$

$$\Rightarrow (a)^{\frac{1}{3}}\sin\theta = (b)^{\frac{1}{3}}\cos\theta$$

$$\Rightarrow \tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$$

$$\therefore \sin\theta = \frac{(b)^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}} \text{ and } \cos\theta = \frac{(a)^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}} \qquad ...(2)$$

It can be clearly shown that $\frac{d^2(AC)}{d\theta^2} < 0$ when $\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$.

Therefore, by second derivative test, the length of the hypotenuse is the maximum when

$\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$.

Now, when $\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$, we have:

$$AC = \frac{b\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}}{b^{\frac{1}{3}}} + \frac{a\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}}{a^{\frac{1}{3}}} \qquad \text{[Using (1) and (2)]}$$

$$= \sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}\left(b^{\frac{2}{3}} + a^{\frac{2}{3}}\right)$$

$$= \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$$

Hence, the maximum length of the hypotenuses is $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$.

Question 13:

Find the points at which the function *f* given by $f(x) = (x-2)^4 (x+1)^3$ has

(i) local maxima (ii) local minima

(iii) point of inflexion

Answer

The given function is $f(x) = (x-2)^4 (x+1)^3$.

$$\therefore f'(x) = 4(x-2)^3 (x+1)^3 + 3(x+1)^2 (x-2)^4$$

$$= (x-2)^3 (x+1)^2 \left[4(x+1) + 3(x-2)\right]$$

$$= (x-2)^3 (x+1)^2 (7x-2)$$

$$\text{Now, } f'(x) = 0 \Rightarrow x = -1 \text{ and } x = \frac{2}{7} \text{ or } x = 2$$

Now, for values of *x* close to $\frac{2}{7}$ and to the left of $\frac{2}{7}, f'(x) > 0.$ Also, for values of *x* close to $\frac{2}{7}$ and to the right of $\frac{2}{7}, f'(x) < 0.$

Thus, $x = \frac{2}{7}$ is the point of local maxima.

Now, for values of *x* close to 2 and to the left of $2, f'(x) < 0.$ Also, for values of *x* close to 2 and to the right of 2, $f'(x) > 0.$

Thus, *x* = 2 is the point of local minima.

Now, as the value of *x* varies through −1, $f'(x)$ does not changes its sign.

Thus, *x* = −1 is the point of inflexion.

Question 14:

Find the absolute maximum and minimum values of the function *f* given by

$f(x) = \cos^2 x + \sin x, x \in [0, \pi]$

Answer

$$f(x) = \cos^2 x + \sin x$$

$$f'(x) = 2\cos x(-\sin x) + \cos x$$

$$= -2\sin x \cos x + \cos x$$

$$\text{Now, } f'(x) = 0$$

$$\Rightarrow 2\sin x \cos x = \cos x \Rightarrow \cos x(2\sin x - 1) = 0$$

$$\Rightarrow \sin x = \frac{1}{2} \text{ or } \cos x = 0$$

$$\Rightarrow x = \frac{\pi}{6}, \text{ or } \frac{\pi}{2} \text{ as } x \in [0, \pi]$$

Now, evaluating the value of *f* at critical points $x = \frac{\pi}{2}$ and $x = \frac{\pi}{6}$ and at the end points of the interval $[0, \pi]$ (i.e., at *x* = 0 and *x* = π), we have:

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$$f\left(\frac{\pi}{6}\right)=\cos^2\frac{\pi}{6}+\sin\frac{\pi}{6}=\left(\frac{\sqrt{3}}{2}\right)^2+\frac{1}{2}=\frac{5}{4}$$

$$f(0)=\cos^2 0+\sin 0=1+0=1$$

$$f(\pi)=\cos^2\pi+\sin\pi=(-1)^2+0=1$$

$$f\left(\frac{\pi}{2}\right)=\cos^2\frac{\pi}{2}+\sin\frac{\pi}{2}=0+1=1$$

Hence, the absolute maximum value of *f* is $\frac{5}{4}$ occurring at $x=\frac{\pi}{6}$ and the absolute minimum value of *f* is 1 occurring at $x=0,\frac{\pi}{2},\text{and}\pi.$

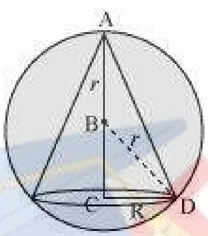
Question 15:

Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius *r* is $\frac{4r}{3}$.

Answer

A sphere of fixed radius (*r*) is given.

Let *R* and *h* be the radius and the height of the cone respectively.



The volume (*V*) of the cone is given by,

$$V=\frac{1}{3}\pi R^2 h$$

Now, from the right triangle BCD, we have:

$$BC=\sqrt{r^2-R^2}$$

$\square h=r+\sqrt{r^2-R^2}$

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$$\therefore V = \frac{1}{3}\pi R^2\left(r + \sqrt{r^2 - R^2}\right) = \frac{1}{3}\pi R^2 r + \frac{1}{3}\pi R^2\sqrt{r^2 - R^2}$$

$$\therefore \frac{dV}{dR} = \frac{2}{3}\pi Rr + \frac{2\pi}{3}\pi R\sqrt{r^2 - R^2} + \frac{R^2}{3}\cdot\frac{(-2R)}{2\sqrt{r^2 - R^2}}$$

$$= \frac{2}{3}\pi Rr + \frac{2\pi}{3}\pi R\sqrt{r^2 - R^2} - \frac{R^3}{3\sqrt{r^2 - R^2}}$$

$$= \frac{2}{3}\pi Rr + \frac{2\pi R\left(r^2 - R^2\right) - \pi R^3}{3\sqrt{r^2 - R^2}}$$

$$= \frac{2}{3}\pi Rr + \frac{2\pi Rr^2 - 3\pi R^3}{3\sqrt{r^2 - R^2}}$$

Now, $\frac{dV}{dR^2} = 0$

$$\Rightarrow \frac{2\pi rR}{3} = \frac{3\pi R^3 - 2\pi Rr^2}{3\sqrt{r^2 - R^2}}$$

$$\Rightarrow 2r\sqrt{r^2 - R^2} = 3R^2 - 2r^2$$

$$\Rightarrow 4r^2\left(r^2 - R^2\right) = \left(3R^2 - 2r^2\right)^2$$

$$\Rightarrow 4r^4 - 4r^2R^2 = 9R^4 + 4r^4 - 12R^2r^2$$

$$\Rightarrow 9R^4 - 8r^2R^2 = 0$$

$$\Rightarrow 9R^2 = 8r^2$$

$$\Rightarrow R^2 = \frac{8r^2}{9}$$

Now, $\frac{d^2V}{dR^2} = \frac{2\pi r}{3} + \frac{3\sqrt{r^2 - R^2}\left(2\pi r^2 - 9\pi R^2\right) - \left(2\pi Rr^2 - 3\pi R^3\right)(-6R)\frac{1}{2\sqrt{r^2 - R^2}}}{9\left(r^2 - R^2\right)}$

$$= \frac{2\pi r}{3} + \frac{3\sqrt{r^2 - R^2}\left(2\pi r^2 - 9\pi R^2\right) + \left(2\pi Rr^2 - 3\pi R^3\right)(3R)\frac{1}{2\sqrt{r^2 - R^2}}}{9\left(r^2 - R^2\right)}$$

Now, when $R^2 = \frac{8r^2}{9}$, it can be shown that $\frac{d^2V}{dR^2} < 0$.

☐ The volume is the maximum when $R^2 = \frac{8r^2}{9}$.

When $R^2 = \frac{8r^2}{9}$, height of the cone $= r + \sqrt{r^2 - \frac{8r^2}{9}} = r + \sqrt{\frac{r^2}{9}} = r + \frac{r}{3} = \frac{4r}{3}$.

Hence, it can be seen that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius *r* is $\frac{4r}{3}$.

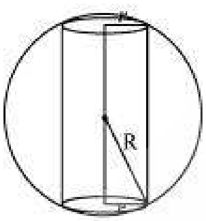
Question 17:

Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius *R* is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

Answer

A sphere of fixed radius (*R)* is given.

Let *r* and *h* be the radius and the height of the cylinder respectively.



From the given figure, we have $h = 2\sqrt{R^2 - r^2}$.

The volume (*V)* of the cylinder is given by,

$$V = \pi r^2 h = 2\pi r^2 \sqrt{R^2 - r^2}$$

$$\therefore \frac{dV}{dr} = 4\pi r\sqrt{R^2 - r^2} + \frac{2\pi r^2(-2r)}{2\sqrt{R^2 - r^2}}$$

$$= 4\pi r\sqrt{R^2 - r^2} - \frac{2\pi r^3}{\sqrt{R^2 - r^2}}$$

$$= \frac{4\pi r(R^2 - r^2) - 2\pi r^3}{\sqrt{R^2 - r^2}}$$

$$= \frac{4\pi r R^2 - 6\pi r^3}{\sqrt{R^2 - r^2}}$$

$$\text{Now, } \frac{dV}{dr} = 0 \Rightarrow 4\pi r R^2 - 6\pi r^3 = 0$$

$$\Rightarrow r^2 = \frac{2R^2}{3}$$

$$\text{Now, } \frac{d^2V}{dr^2} = \frac{\sqrt{R^2 - r^2}(4\pi R^2 - 18\pi r^2) - (4\pi r R^2 - 6\pi r^3)\frac{(-2r)}{2\sqrt{R^2 - r^2}}}{(R^2 - r^2)}$$

$$= \frac{(R^2 - r^2)(4\pi R^2 - 18\pi r^2) + r(4\pi r R^2 - 6\pi r^3)}{(R^2 - r^2)^{\frac{3}{2}}}$$

$$= \frac{4\pi R^4 - 22\pi r^2 R^2 + 12\pi r^4 + 4\pi r^2 R^2}{(R^2 - r^2)^{\frac{3}{2}}}$$

Now, it can be observed that at $r^2 = \frac{2R^2}{3}, \frac{d^2V}{dr^2} < 0$.

□The volume is the maximum when $r^2 = \frac{2R^2}{3}$.

When $r^2 = \frac{2R^2}{3}$, the height of the cylinder is $2\sqrt{R^2 - \frac{2R^2}{3}} = 2\sqrt{\frac{R^2}{3}} = \frac{2R}{\sqrt{3}}$.

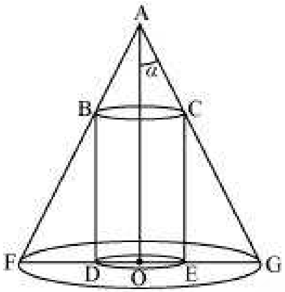
Hence, the volume of the cylinder is the maximum when the height of the cylinder is $\frac{2R}{\sqrt{3}}$.

Question 18:

Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height *h* and semi vertical angle *α* is one-third that of the cone and the greatest volume of cylinder is $\frac{4}{27}\pi h^3 \tan^2 \alpha$.

Answer

The given right circular cone of fixed height (*h*) and semi-vertical angle (*α*) can be drawn as:



Here, a cylinder of radius *R* and height *H* is inscribed in the cone.

Then, ☐GAO = *α*, OG = *r*, OA = *h*, OE = *R*, and CE = *H*.

We have,

$r = h \tan \alpha$

Now, since ΔAOG is similar to ΔCEG, we have:

$$\frac{AO}{OG} = \frac{CE}{EG}$$

$$\Rightarrow \frac{h}{r} = \frac{H}{r-R} \qquad [EG = OG - OE]$$

$$\Rightarrow H = \frac{h}{r}(r-R) = \frac{h}{h\tan\alpha}(h\tan\alpha - R) = \frac{1}{\tan\alpha}(h\tan\alpha - R)$$

Now, the volume (*V*) of the cylinder is given by,

$$V = \pi R^2 H = \frac{\pi R^2}{\tan\alpha}(h\tan\alpha - R) = \pi R^2 h - \frac{\pi R^3}{\tan\alpha}$$

$$\therefore \frac{dV}{dR} = 2\pi R h - \frac{3\pi R^2}{\tan\alpha}$$

$$\text{Now, } \frac{dV}{dR} = 0$$

$$\Rightarrow 2\pi Rh = \frac{3\pi R^2}{\tan\alpha}$$

$$\Rightarrow 2h\tan\alpha = 3R$$

$$\Rightarrow R = \frac{2h}{3}\tan\alpha$$

$$\text{Now, } \frac{d^2V}{dR^2} = 2\pi h - \frac{6\pi R}{\tan\alpha}$$

And, for $R = \frac{2h}{3}\tan\alpha$, we have:

$$\frac{d^2V}{dR^2} = 2\pi h - \frac{6\pi}{\tan\alpha}\left(\frac{2h}{3}\tan\alpha\right) = 2\pi h - 4\pi h = -2\pi h < 0$$

□By second derivative test, the volume of the cylinder is the greatest when

$$R = \frac{2h}{3}\tan\alpha.$$

$$\text{When } R = \frac{2h}{3}\tan\alpha,\ H = \frac{1}{\tan\alpha}\left(h\tan\alpha - \frac{2h}{3}\tan\alpha\right) = \frac{1}{\tan\alpha}\left(\frac{h\tan\alpha}{3}\right) = \frac{h}{3}.$$

Thus, the height of the cylinder is one-third the height of the cone when the volume of the cylinder is the greatest.

Now, the maximum volume of the cylinder can be obtained as:

$$\pi\left(\frac{2h}{3}\tan\alpha\right)^2\left(\frac{h}{3}\right) = \pi\left(\frac{4h^2}{9}\tan^2\alpha\right)\left(\frac{h}{3}\right) = \frac{4}{27}\pi h^3\tan^2\alpha$$

Hence, the given result is proved.

Question 19:

A cylindrical tank of radius 10 m is being filled with wheat at the rate of 314 cubic mere per hour. Then the depth of the wheat is increasing at the rate of

(A) 1 m/h (B) 0.1 m/h

(C) 1.1 m/h (D) 0.5 m/h

Answer

Let r be the radius of the cylinder.

Then, volume (V) of the cylinder is given by,

$$V = \pi(\text{radius})^2 \times \text{height}$$
$$= \pi(10)^2 h \qquad (\text{radius} = 10\text{ m})$$
$$= 100\pi h$$

Differentiating with respect to time t, we have:

$$\frac{dV}{dt} = 100\pi \frac{dh}{dt}$$

The tank is being filled with wheat at the rate of 314 cubic metres per hour.

$$\therefore \frac{dV}{dt} = 314\text{ m}^3/\text{h}$$

Thus, we have:

$$314 = 100\pi \frac{dh}{dt}$$
$$\Rightarrow \frac{dh}{dt} = \frac{314}{100(3.14)} = \frac{314}{314} = 1$$

Hence, the depth of wheat is increasing at the rate of 1 m/h.

The correct answer is A.

Question 20:

The slope of the tangent to the curve $x = t^2 + 3t - 8, y = 2t^2 - 2t - 5$ at the point (2, −1) is

(A) $\frac{22}{7}$ (B) $\frac{6}{7}$ (C) $\frac{7}{6}$ (D) $\frac{-6}{7}$

Answer

The given curve is $x = t^2 + 3t - 8$ and $y = 2t^2 - 2t - 5$.

$$\therefore \frac{dx}{dt} = 2t + 3 \text{ and } \frac{dy}{dt} = 4t - 2$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{4t-2}{2t+3}$$

The given point is (2, −1).

At $x = 2$, we have:

$t^2 + 3t - 8 = 2$

$\Rightarrow t^2 + 3t - 10 = 0$

$\Rightarrow (t-2)(t+5) = 0$

$\Rightarrow t = 2 \text{ or } t = -5$

At $y = -1$, we have:

$2t^2 - 2t - 5 = -1$

$\Rightarrow 2t^2 - 2t - 4 = 0$

$\Rightarrow 2(t^2 - t - 2) = 0$

$\Rightarrow (t-2)(t+1) = 0$

$\Rightarrow t = 2 \text{ or } t = -1$

The common value of t is 2.

Hence, the slope of the tangent to the given curve at point (2, −1) is

$$\left.\frac{dy}{dx}\right]_{t=2} = \frac{4(2)-2}{2(2)+3} = \frac{8-2}{4+3} = \frac{6}{7}.$$

The correct answer is B.

Question 21:

The line $y = mx + 1$ is a tangent to the curve $y^2 = 4x$ if the value of m is

(A) 1 (B) 2 (C) 3 (D) $\frac{1}{2}$

Answer

The equation of the tangent to the given curve is $y = mx + 1$.

Now, substituting $y = mx + 1$ in $y^2 = 4x$, we get:

$\Rightarrow (mx+1)^2 = 4x$

$\Rightarrow m^2x^2 + 1 + 2mx - 4x = 0$

$\Rightarrow m^2x^2 + x(2m-4) + 1 = 0 \qquad \text{...(i)}$

Since a tangent touches the curve at one point, the roots of equation (i) must be equal.

Therefore, we have:

$\text{Discriminant} = 0$

$(2m-4)^2 - 4(m^2)(1) = 0$

$\Rightarrow 4m^2 + 16 - 16m - 4m^2 = 0$

$\Rightarrow 16 - 16m = 0$

$\Rightarrow m = 1$

Hence, the required value of m is 1.

The correct answer is A.

Question 22:

The normal at the point (1, 1) on the curve $2y + x^2 = 3$ is

(A) $x + y = 0$ (B) $x - y = 0$

(C) $x + y + 1 = 0$ (D) $x - y = 1$

Answer

The equation of the given curve is $2y + x^2 = 3$.

Differentiating with respect to x, we have:

$$\frac{2dy}{dx} + 2x = 0$$

$$\Rightarrow \frac{dy}{dx} = -x$$

$$\therefore \left.\frac{dy}{dx}\right]_{(1,1)} = -1$$

The slope of the normal to the given curve at point (1, 1) is

$$\frac{-1}{\left.\frac{dy}{dx}\right]_{(1,1)}} = 1.$$

Hence, the equation of the normal to the given curve at (1, 1) is given as:

$\Rightarrow y - 1 = 1(x-1)$

$\Rightarrow y - 1 = x - 1$

$\Rightarrow x - y = 0$

The correct answer is B.

Question 23:

The normal to the curve $x^2 = 4y$ passing (1, 2) is

(A) $x + y = 3$ (B) $x - y = 3$

(C) $x + y = 1$ (D) $x - y = 1$

Answer

The equation of the given curve is $x^2 = 4y$.

Differentiating with respect to x, we have:

$$2x = 4 \cdot \frac{dy}{dx}$$
$$\Rightarrow \frac{dy}{dx} = \frac{x}{2}$$

The slope of the normal to the given curve at point (h, k) is given by,

$$\frac{-1}{\left.\frac{dy}{dx}\right]_{(h,k)}} = -\frac{2}{h}$$

□Equation of the normal at point (h, k) is given as:

$$y - k = \frac{-2}{h}(x - h)$$

Now, it is given that the normal passes through the point (1, 2).

Therefore, we have:

$$2 - k = \frac{-2}{h}(1 - h) \text{ or } k = 2 + \frac{2}{h}(1 - h) \quad \text{... (i)}$$

Since (h, k) lies on the curve $x^2 = 4y$, we have $h^2 = 4k$.

$$\Rightarrow k = \frac{h^2}{4}$$

From equation (i), we have:

$$\frac{h^2}{4} = 2 + \frac{2}{h}(1-h)$$
$$\Rightarrow \frac{h^3}{4} = 2h + 2 - 2h = 2$$
$$\Rightarrow h^3 = 8$$
$$\Rightarrow h = 2$$
$$\therefore k = \frac{h^2}{4} \Rightarrow k = 1$$

Hence, the equation of the normal is given as:

$$\Rightarrow y - 1 = \frac{-2}{2}(x-2)$$
$$\Rightarrow y - 1 = -(x-2)$$
$$\Rightarrow x + y = 3$$

The correct answer is A.

Question 24:

The points on the curve $9y^2 = x^3$, where the normal to the curve makes equal intercepts with the axes are

(A) $\left(4, \pm\frac{8}{3}\right)$ (B) $\left(4, \frac{-8}{3}\right)$

(C) $\left(4, \pm\frac{3}{8}\right)$ (D) $\left(\pm 4, \frac{3}{8}\right)$

Answer

The equation of the given curve is $9y^2 = x^3$.

Differentiating with respect to x, we have:

$$9(2y)\frac{dy}{dx} = 3x^2$$
$$\Rightarrow \frac{dy}{dx} = \frac{x^2}{6y}$$

The slope of the normal to the given curve at point (x_1, y_1) is

$$\frac{-1}{\left.\frac{dy}{dx}\right]_{(x_1,y_1)}} = -\frac{6y_1}{x_1^2}.$$

☐ The equation of the normal to the curve at (x_1, y_1) is

$$y - y_1 = \frac{-6y_1}{x_1^2}(x - x_1).$$

$$\Rightarrow x_1^2 y - x_1^2 y_1 = -6xy_1 + 6x_1 y_1$$

$$\Rightarrow 6xy_1 + x_1^2 y = 6x_1 y_1 + x_1^2 y_1$$

$$\Rightarrow \frac{6xy_1}{6x_1 y_1 + x_1^2 y_1} + \frac{x_1^2 y}{6x_1 y_1 + x_1^2 y_1} = 1$$

$$\Rightarrow \frac{x}{\frac{x_1(6 + x_1)}{6}} + \frac{y}{\frac{y_1(6 + x_1)}{x_1}} = 1$$

It is given that the normal makes equal intercepts with the axes.

Therefore, We have:

$$\therefore \frac{x_1(6 + x_1)}{6} = \frac{y_1(6 + x_1)}{x_1}$$

$$\Rightarrow \frac{x_1}{6} = \frac{y_1}{x_1}$$

$$\Rightarrow x_1^2 = 6y_1 \qquad \text{...(i)}$$

Also, the point (x_1, y_1) lies on the curve, so we have

$$9y_1^2 = x_1^3 \qquad \text{...(ii)}$$

From (i) and (ii), we have:

$$9\left(\frac{x_1^2}{6}\right)^2 = x_1^3 \Rightarrow \frac{x_1^4}{4} = x_1^3 \Rightarrow x_1 = 4$$

From (ii), we have:

$9y_1^2 = (4)^3 = 64$

$\Rightarrow y_1^2 = \frac{64}{9}$

$\Rightarrow y_1 = \pm\frac{8}{3}$

Hence, the required points are $\left(4, \pm\frac{8}{3}\right)$.

The correct answer is A.

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