Exercise 2.2

Question 1:

Prove $3\sin^{-1}x = \sin^{-1}\left(3x - 4x^3\right),\ x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

Answer

To prove: $3\sin^{-1}x = \sin^{-1}\left(3x - 4x^3\right),\ x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

Let x = sinθ. Then, $\sin^{-1}x = \theta.$

We have,

R.H.S. = $\sin^{-1}\left(3x - 4x^3\right) = \sin^{-1}\left(3\sin\theta - 4\sin^3\theta\right)$

$= \sin^{-1}\left(\sin 3\theta\right)$

= 3θ

$= 3\sin^{-1}x$

= L.H.S.

Question 2:

Prove $3\cos^{-1}x = \cos^{-1}\left(4x^3 - 3x\right),\ x \in \left[\frac{1}{2}, 1\right]$

Answer

To prove: $3\cos^{-1}x = \cos^{-1}\left(4x^3 - 3x\right),\ x \in \left[\frac{1}{2}, 1\right]$

Let x = cosθ. Then, $\cos^{-1}x = \theta$.

We have,

$$\text{R.H.S.} = \cos^{-1}\left(4x^3 - 3x\right)$$
$$= \cos^{-1}\left(4\cos^3\theta - 3\cos\theta\right)$$
$$= \cos^{-1}\left(\cos 3\theta\right)$$
$$= 3\theta$$
$$= 3\cos^{-1}x$$
$$= \text{L.H.S.}$$

Question 3:

Prove $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$

Answer

To prove: $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$

$$\text{L.H.S.} = \tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24}$$
$$= \tan^{-1}\frac{\frac{2}{11} + \frac{7}{24}}{1 - \frac{2}{11}\cdot\frac{7}{24}} \qquad \left[\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$
$$= \tan^{-1}\frac{\frac{48+77}{11\times 24}}{\frac{11\times 24 - 14}{11\times 24}}$$
$$= \tan^{-1}\frac{48+77}{264-14} = \tan^{-1}\frac{125}{250} = \tan^{-1}\frac{1}{2} = \text{R.H.S.}$$

Question 4:

Prove $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$

Answer

To prove: $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$

$$\text{L.H.S.} = 2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{2\cdot\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2} + \tan^{-1}\frac{1}{7} \qquad \left[2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$= \tan^{-1}\frac{1}{\left(\frac{3}{4}\right)} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{4}{3} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\cdot\frac{1}{7}} \qquad \left[\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\frac{\left(\frac{28+3}{21}\right)}{\left(\frac{21-4}{21}\right)}$$

$$= \tan^{-1}\frac{31}{17} = \text{R.H.S.}$$

Question 5:

Write the function in the simplest form:

$$\tan^{-1}\frac{\sqrt{1+x^2}-1}{x},\ x \neq 0$$

Answer

$$\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$$

Put $x=\tan\theta \Rightarrow \theta=\tan^{-1}x$

$$\therefore \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}=\tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$=\tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$

$$=\tan^{-1}\left(\tan\frac{\theta}{2}\right)=\frac{\theta}{2}=\frac{1}{2}\tan^{-1}x$$

Question 6:

Write the function in the simplest form:

$$\tan^{-1}\frac{1}{\sqrt{x^2-1}},\ |x|>1$$

Answer

$$\tan^{-1}\frac{1}{\sqrt{x^2-1}},\ |x|>1$$

Put x = cosec $\theta \Rightarrow \theta$ = cosec^{-1} x

$$\therefore \tan^{-1}\frac{1}{\sqrt{x^2-1}}=\tan^{-1}\frac{1}{\sqrt{\cos ec^2\theta-1}}$$

$$=\tan^{-1}\left(\frac{1}{\cot\theta}\right)=\tan^{-1}(\tan\theta)$$

$$=\theta=\operatorname{cosec}^{-1}x=\frac{\pi}{2}-\sec^{-1}x \qquad \left[\operatorname{cosec}^{-1}x+\sec^{-1}x=\frac{\pi}{2}\right]$$

Question 7:

Write the function in the simplest form:

$$\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right),\ x<\pi$$

Answer

$$\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right),\ x<\pi$$

$$\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)=\tan^{-1}\left(\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}\right)$$

$$=\tan^{-1}\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)=\tan^{-1}\left(\tan\frac{x}{2}\right)$$

$$=\frac{x}{2}$$

Question 8:

Write the function in the simplest form:

$$\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right),\ 0<x<\pi$$

Answer

$$\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)$$

$$=\tan^{-1}\left(\frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}}\right)$$

$$=\tan^{-1}\left(\frac{1-\tan x}{1+\tan x}\right)$$

$$=\tan^{-1}(1)-\tan^{-1}(\tan x) \qquad \left[\tan^{-1}\frac{x-y}{1-xy}=\tan^{-1}x-\tan^{-1}y\right]$$

$$=\frac{\pi}{4}-x$$

Question 9:

Write the function in the simplest form:

$$\tan^{-1}\frac{x}{\sqrt{a^2-x^2}},\ |x|<a$$

Answer

$$\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}$$

$$\text{Put } x=a\sin\theta \Rightarrow \frac{x}{a}=\sin\theta \Rightarrow \theta=\sin^{-1}\left(\frac{x}{a}\right)$$

$$\therefore \tan^{-1}\frac{x}{\sqrt{a^2-x^2}}=\tan^{-1}\left(\frac{a\sin\theta}{\sqrt{a^2-a^2\sin^2\theta}}\right)$$

$$=\tan^{-1}\left(\frac{a\sin\theta}{a\sqrt{1-\sin^2\theta}}\right)=\tan^{-1}\left(\frac{a\sin\theta}{a\cos\theta}\right)$$

$$=\tan^{-1}(\tan\theta)=\theta=\sin^{-1}\frac{x}{a}$$

Question 10:

Write the function in the simplest form:

$$\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right),\ a>0;\ \frac{-a}{\sqrt{3}}\le x\le\frac{a}{\sqrt{3}}$$

Answer

$$\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right)$$

$$\text{Put } x=a\tan\theta \Rightarrow \frac{x}{a}=\tan\theta \Rightarrow \theta=\tan^{-1}\frac{x}{a}$$

$$\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right)=\tan^{-1}\left(\frac{3a^2\cdot a\tan\theta-a^3\tan^3\theta}{a^3-3a\cdot a^2\tan^2\theta}\right)$$

$$=\tan^{-1}\left(\frac{3a^3\tan\theta-a^3\tan^3\theta}{a^3-3a^3\tan^2\theta}\right)$$

$$=\tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$$

$$=\tan^{-1}(\tan 3\theta)$$

$$=3\theta$$

$$=3\tan^{-1}\frac{x}{a}$$

Question 11:

Find the value of $\tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]$

Answer

Let $\sin^{-1}\frac{1}{2}=x$. Then, $\sin x=\frac{1}{2}=\sin\left(\frac{\pi}{6}\right)$.

$$\therefore \sin^{-1}\frac{1}{2}=\frac{\pi}{6}$$

$$\therefore \tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]=\tan^{-1}\left[2\cos\left(2\times\frac{\pi}{6}\right)\right]$$

$$=\tan^{-1}\left[2\cos\frac{\pi}{3}\right]=\tan^{-1}\left[2\times\frac{1}{2}\right]$$

$$=\tan^{-1}1=\frac{\pi}{4}$$

Question 12:

Find the value of $\cot\left(\tan^{-1} a+\cot^{-1} a\right)$

Answer

$$\cot\left(\tan^{-1} a+\cot^{-1} a\right)$$

$$=\cot\left(\frac{\pi}{2}\right) \qquad \left[\tan^{-1} x+\cot^{-1} x=\frac{\pi}{2}\right]$$

$$=0$$

Question 13:

Find the value of $\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right],\ |x|<1,\ y>0 \text{ and } xy<1$

Answer

Let $x = \tan\theta$. Then, $\theta = \tan^{-1} x$.

$$\therefore \sin^{-1}\frac{2x}{1+x^2}=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)=\sin^{-1}(\sin 2\theta)=2\theta=2\tan^{-1} x$$

Let $y = \tan\Phi$. Then, $\Phi = \tan^{-1} y$.

$$\therefore \cos^{-1}\frac{1-y^2}{1+y^2}=\cos^{-1}\left(\frac{1-\tan^2\phi}{1+\tan^2\phi}\right)=\cos^{-1}(\cos 2\phi)=2\phi=2\tan^{-1} y$$

$$\therefore \tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right]$$

$$=\tan\frac{1}{2}\left[2\tan^{-1} x+2\tan^{-1} y\right]$$

$$=\tan\left[\tan^{-1} x+\tan^{-1} y\right]$$

$$=\tan\left[\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$=\frac{x+y}{1-xy}$$

Question 14:

If $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$, then find the value of x.

Answer

$$\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$$

$$\Rightarrow \sin\left(\sin^{-1}\frac{1}{5}\right)\cos\left(\cos^{-1}x\right)+\cos\left(\sin^{-1}\frac{1}{5}\right)\sin\left(\cos^{-1}x\right)=1$$

$$\left[\sin(A+B)=\sin A\cos B+\cos A\sin B\right]$$

$$\Rightarrow \frac{1}{5}\times x+\cos\left(\sin^{-1}\frac{1}{5}\right)\sin\left(\cos^{-1}x\right)=1$$

$$\Rightarrow \frac{x}{5}+\cos\left(\sin^{-1}\frac{1}{5}\right)\sin\left(\cos^{-1}x\right)=1 \qquad ...(1)$$

Now, let $\sin^{-1}\frac{1}{5}=y$.

Then, $\sin y=\frac{1}{5}\Rightarrow \cos y=\sqrt{1-\left(\frac{1}{5}\right)^2}=\frac{2\sqrt{6}}{5}\Rightarrow y=\cos^{-1}\left(\frac{2\sqrt{6}}{5}\right)$.

$$\therefore \sin^{-1}\frac{1}{5}=\cos^{-1}\left(\frac{2\sqrt{6}}{5}\right) \qquad ...(2)$$

Let $\cos^{-1}x=z$.

Then, $\cos z=x\Rightarrow \sin z=\sqrt{1-x^2}\Rightarrow z=\sin^{-1}\left(\sqrt{1-x^2}\right)$.

$$\therefore \cos^{-1}x=\sin^{-1}\left(\sqrt{1-x^2}\right) \qquad ...(3)$$

From (1), (2), and (3) we have:

$$\frac{x}{5}+\cos\left(\cos^{-1}\frac{2\sqrt{6}}{5}\right)\cdot\sin\left(\sin^{-1}\sqrt{1-x^2}\right)=1$$

$$\Rightarrow \frac{x}{5}+\frac{2\sqrt{6}}{5}\cdot\sqrt{1-x^2}=1$$

$$\Rightarrow x+2\sqrt{6}\sqrt{1-x^2}=5$$

$$\Rightarrow 2\sqrt{6}\sqrt{1-x^2}=5-x$$

On squaring both sides, we get:

$$(4)(6)(1-x^2)=25+x^2-10x$$
$$\Rightarrow 24-24x^2=25+x^2-10x$$
$$\Rightarrow 25x^2-10x+1=0$$
$$\Rightarrow (5x-1)^2=0$$
$$\Rightarrow (5x-1)=0$$
$$\Rightarrow x=\frac{1}{5}$$

Hence, the value of *x* is $\frac{1}{5}$.

Question 15:

If $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$, then find the value of *x*.

Answer

$$\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right]=\frac{\pi}{4} \qquad \left[\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right]$$

$$\Rightarrow \tan^{-1}\left[\frac{(x-1)(x+2)+(x+1)(x-2)}{(x+2)(x-2)-(x-1)(x+1)}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{x^2+x-2+x^2-x-2}{x^2-4-x^2+1}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{2x^2-4}{-3}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan\left[\tan^{-1}\frac{4-2x^2}{3}\right]=\tan\frac{\pi}{4}$$

$$\Rightarrow \frac{4-2x^2}{3}=1$$

$$\Rightarrow 4-2x^2=3$$

$$\Rightarrow 2x^2=4-3=1$$

$$\Rightarrow x=\pm\frac{1}{\sqrt{2}}$$

Hence, the value of *x* is $\pm\frac{1}{\sqrt{2}}$.

Question 16:

Find the values of $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

Answer

$$\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$$

We know that $\sin^{-1}(\sin x) = x$ if $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, which is the principal value branch of $\sin^{-1}x$.

Here, $\frac{2\pi}{3} \notin \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

Now, $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ can be written as:

$$\sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \sin^{-1}\left[\sin\left(\pi - \frac{2\pi}{3}\right)\right] = \sin^{-1}\left(\sin\frac{\pi}{3}\right) \text{ where } \frac{\pi}{3} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$$

$$\therefore \sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \sin^{-1}\left(\sin\frac{\pi}{3}\right) = \frac{\pi}{3}$$

Question 17:

Find the values of $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

Answer

$$\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$$

We know that $\tan^{-1}(\tan x) = x$ if $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, which is the principal value branch of $\tan^{-1}x$.

Here, $\frac{3\pi}{4} \notin \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

Now, $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$ can be written as:

$$\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \tan^{-1}\left[-\tan\left(\frac{-3\pi}{4}\right)\right] = \tan^{-1}\left[-\tan\left(\pi - \frac{\pi}{4}\right)\right]$$

$$= \tan^{-1}\left[-\tan\frac{\pi}{4}\right] = \tan^{-1}\left[\tan\left(-\frac{\pi}{4}\right)\right] \textit{ where } -\frac{\pi}{4} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

$$\therefore \tan^{-1}\left(\tan\frac{3\pi}{4}\right)=\tan^{-1}\left[\tan\left(\frac{-\pi}{4}\right)\right]=\frac{-\pi}{4}$$

Question 18:

Find the values of $\tan\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)$

Answer

Let $\sin^{-1}\frac{3}{5}=x$. Then, $\sin x=\frac{3}{5}\Rightarrow \cos x=\sqrt{1-\sin^2 x}=\frac{4}{5}\Rightarrow \sec x=\frac{5}{4}$.

$$\therefore \tan x=\sqrt{\sec^2 x-1}=\sqrt{\frac{25}{16}-1}=\frac{3}{4}$$

$$\therefore x=\tan^{-1}\frac{3}{4}$$

$$\therefore \sin^{-1}\frac{3}{5}=\tan^{-1}\frac{3}{4} \qquad \text{...(i)}$$

$$\text{Now, } \cot^{-1}\frac{3}{2}=\tan^{-1}\frac{2}{3} \qquad \text{...(ii)} \qquad \left[\tan^{-1}\frac{1}{x}=\cot^{-1}x\right]$$

$$\text{Hence, } \tan\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)$$

$$=\tan\left(\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{2}{3}\right) \qquad [\text{Using (i) and (ii)}]$$

$$=\tan\left(\tan^{-1}\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\cdot\frac{2}{3}}\right) \qquad \left[\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right]$$

$$=\tan\left(\tan^{-1}\frac{9+8}{12-6}\right)$$

$$=\tan\left(\tan^{-1}\frac{17}{6}\right)=\frac{17}{6}$$

Question 19:

Find the values of $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ is equal to

(A) $\frac{7\pi}{6}$ **(B)** $\frac{5\pi}{6}$ **(C)** $\frac{\pi}{3}$ **(D)** $\frac{\pi}{6}$

Answer

We know that $\cos^{-1}(\cos x) = x$ if $x \in [0, \pi]$, which is the principal value branch of $\cos^{-1}x$.

Here, $\frac{7\pi}{6} \notin x \in [0, \pi]$.

Now, $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ can be written as:

$$\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left(\cos\frac{-7\pi}{6}\right) = \cos^{-1}\left[\cos\left(2\pi - \frac{7\pi}{6}\right)\right] \quad [\cos(2\pi + x) = \cos x]$$

$$= \cos^{-1}\left[\cos\frac{5\pi}{6}\right] \text{ where } \frac{5\pi}{6} \in [0, \pi]$$

$$\therefore \cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left(\cos\frac{5\pi}{6}\right) = \frac{5\pi}{6}$$

The correct answer is B.

Question 20:

Find the values of $\sin\left(\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right)$ is equal to

(A) $\frac{1}{2}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{4}$ **(D)** 1

Answer

Let $\sin^{-1}\left(\frac{-1}{2}\right) = x$. Then, $\sin x = \frac{-1}{2} = -\sin\frac{\pi}{6} = \sin\left(\frac{-\pi}{6}\right)$.

We know that the range of the principal value branch of $\sin^{-1}$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

$\therefore \sin^{-1}\left(\frac{-1}{2}\right)=\frac{-\pi}{6}$

$$\therefore \sin\left(\frac{\pi}{3}-\sin^{-1}\left(\frac{-1}{2}\right)\right)=\sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right)=\sin\left(\frac{3\pi}{6}\right)=\sin\left(\frac{\pi}{2}\right)=1$$

The correct answer is D.