MISCELLANEOUS

Q1. Let f: R → R be defined as f(x) = 10x + 7. Find the function g: R → R such that g o f = f o g = 1R.

A.1. It is given that $f: R \to R$ is defined as $f(x) = 10x + 7.$

One-one:

$Let,\ f(x) = f(y), where, x, y \in R$

$\Rightarrow 10x + 7 = 10y + 7$

$\Rightarrow x = y$

$\therefore f$ is a one-one function.

Onto:

$For, y \in R, let, y = 10x + 7$

$\Rightarrow x = \dfrac{y-7}{10} \in R$

Therefore, for any $y \in R$,there exists $x = \dfrac{y-7}{10} \in R$

Such that

$$f(x) = f\left(\frac{y-7}{10}\right) = 10\left(\frac{y-7}{10}\right) + 7 = y - 7 + 7 = y$$

$\therefore f$ is onto.

Therefore, f is one-one and onto.

Thus, f is an invertible function.

Let us define $g: R \rightarrow R$ as $g(y) = \dfrac{y-7}{10}$

Now, we have

$$gof(x) = g(f(x)) = g(10x+7) = \frac{(10x+7)-7}{10} = 10\frac{x}{10} = 10$$

And

$$fog(y) = f(g(y)) = f\left(\frac{y-7}{10}\right) = 10\left(\frac{y-7}{10}\right) + 7 = y - 7 + 7 = y$$

Hence, the required function $g: R \rightarrow R$ is defined as $g(y) = \dfrac{y-7}{10}$

Q2. Let f: W → W be defined as f(n) = n − 1, if is odd and f(n) = n + 1, if n is even. Show that f is invertible. Find the inverse of f. Here, W is the set of all whole numbers.

A.2. It is given that:

$f: W \rightarrow W$ is defined as $f(n) = \begin{cases} n-1, if.n, odd \\ n+1, if.n, even \end{cases}$

One-one:

Let, $f(n) = f(m)$.

It can be observed that if n is odd and m is even, then we will have n-1=m+1.

$\Rightarrow n - m = 2$

However, the possibility of n being even and m being odd can also be ignored under a similar argument.

$\therefore$ Both n and m must be either odd or even.

Now, if both n and m are odd, then we have:

$$f(n) = f(m) \Rightarrow n-1 = m-1 \Rightarrow n = m$$

Again, if both n and m are even, then we have:

$$f(n) = f(m) \Rightarrow n+1 = m+1 \Rightarrow n = m$$

$\therefore f$ is one-one.

It is clear that any odd number 2r+1 in co-domain N is the image of 2r in domain N and any even 2r in co-domain N is the image of 2r+1 in domain N.

$\therefore f$ is onto.

Hence, f is an invertible function.

Let us define $g : W \to W$ as:

$$g(m) = \begin{cases} m+1, if.n, even \\ m-1, if.n, odd \end{cases}$$

Now, when n is odd:

$$gof(n) = g(f(n)) = g(n-1) = n-1+1 = n$$

And, when n is even:

$$gof(n) = g(f(n)) = g(n+1) = n+1-1 = n$$

Similarly, when m is odd:

$$fog(m) = f(g(m)) = f(m-1) = m-1+1 = m$$

When m is even:

$$fog(m) = f(g(m)) = f(m+1) = m+1-1 = m$$

$$\therefore gof = I_W, and, fog = I_W$$

Thus, f is invertible and the inverse of f is given by $f^{-1} = g$, which is the same as f.

Hence, the inverse of f is f itself.

Q3. If f: R → R is defined by f(x) = x^2 − 3x + 2, find f(f(x)).

A.3. It is given that $f : R \to R$ is defined as $f(x) = x^2 - 3x + 2$.

$$f\left(f\left(x\right)\right)=f\left(x^2-3x+2\right)$$
$$=\left(x^2+3x+2\right)^2-3\left(x^2-3x+2\right)+2$$
$$=x^4+9x^2+4-6x^2-12x+4x^2-3x^2+9x-6+2$$
$$=x^4-6x^2+10x^2-3x$$

Q4. Show that function f: R → {x ∈ R: −1 < x < 1} defined by f(x) = x/1+|x| x ∈R is one-one and onto function.

A.4. It is given that $f:R\to\{x\in R:-1<x<1\}$ is defined as $f\left(x\right)=\dfrac{x}{1+|x|}, x\in R.$

Suppose $f\left(x\right)=f\left(y\right)$, where $x, y\in R.$

$$\Rightarrow \frac{x}{1+x}=\frac{y}{1-y}\Rightarrow 2xy=x-y$$

Since x is positive and y is negative:

$x>y\Rightarrow x-y>0$

But, 2xy is negative.

Then, $2xy\neq x-y$.

Thus, the case of x being positive and y being negative can be ruled out.

Under a similar argument, x being negative and y being positive can also be ruled out

$\therefore$ x and y have to be either positive or negative.

When x and y are both positive, we have:

$$f\left(x\right)=f\left(y\right)\Rightarrow\frac{x}{1+x}=\frac{y}{1+y}\Rightarrow x+xy=y+xy\Rightarrow x=y$$

When x and y are both negative, we have:

$$f\left(x\right)=f\left(y\right)\Rightarrow\frac{x}{1-x}=\frac{y}{1-y}\Rightarrow x-xy=y-yx\Rightarrow x=y$$

$\therefore f$ is one-one.

Now, let $y\in R$ such that $-1<y<1$.

If x is negative, then there exists $x=\dfrac{y}{1+y}\in R$ such that

$$f(x) = f\left(\frac{y}{1+y}\right) = \frac{\left(\frac{y}{1+y}\right)}{1+\frac{y}{1+y}} = \frac{\frac{y}{1+y}}{1+\frac{-y}{1+y}} = \frac{y}{1+y-y} = y$$

If x is positive, then there exists $x = \frac{y}{1-y} \in R$ such that

$$f(x) = f\left(\frac{y}{1-y}\right) = \frac{\left(\frac{y}{1-y}\right)}{1+\left(\frac{y}{1-y}\right)} = \frac{\frac{y}{1-y}}{1+\frac{y}{1-y}} = \frac{y}{1-y+y} = y$$

$\therefore f$ is onto.

Hence, f is one-one and onto.

Q5. Show that the function f: R → R given by f(x) = x^3 is injective.

A.5. f: R → R is given as f(x) = x^3.

Suppose f(x) = f(y), where x, y ∈ R.

⇒ $x^3 = y^3$ … (1)

Now, we need to show that x = y.

Suppose x ≠ y, their cubes will also not be equal.

⇒ $x^3 \neq y^3$

However, this will be a contradiction to (1).

∴ x = y

Hence, f is injective.

Q6. Give examples of two functions f: N → Z and g: Z → Z such that g o f is injective but g is not injective.

(Hint: Consider f(x) = x and g(x) = |x|)

A.6. Define $f : N \rightarrow Z$ as $f(x)$ and $g : Z \rightarrow Z$ as $g(x) = |x|$

We first show that g is not injective.

It can be observed that:

$g(-1)=|-1|=1$
$g(1)=|1|=1$
$\therefore g(-1)=g(1), but, -1\neq 1$

$\therefore g$ is not injective.

Now, $gof: N\rightarrow Z$ is defined as

$gof(x)=y(f(x))=y(x)=|x|$

Let $x, y\in N$ such that $gof(x)-gof(y)$

$\Rightarrow |x|=|y|$

Since $x, y\in N$, both are positive.

$\therefore |x|=|y|\Rightarrow x=y$

Hence, gof is injective

Q7. Given examples of two functions f: N → N and g: N → N such that gof is onto but f is not onto.

(Hint: Consider f(x) = x + 1 and $g(x)\begin{Bmatrix} x-1 & if, x>1 \\ 1 & f, x=1 \end{Bmatrix}$

A.7. Define $f:\mathrm{N}\rightarrow\mathrm{N}$ by

$f(x)=x+1$

And, $g:\mathrm{N}\rightarrow\mathrm{N}$ by,

$g(x)\begin{Bmatrix} x-1 & if, x>1 \\ 1 & f, x=1 \end{Bmatrix}$

We first show that g is not onto.

For this, consider element 1 in co-domain N. it is clear that this element is not an image of any of the elements in domain.

$\therefore f$ is not onto.

Now, $gof:\mathrm{N}\rightarrow\mathrm{N}$ is defined by,

$gof(x)=g(f(x))=g(x+1)=(x+1)-1[x, in\mathrm{N} \Rightarrow (x+1)>1]$
$=x$

Then, it is clear that for $y \in \mathrm{N}$, there exists $x = y \in \mathrm{N}$ such that $gof(x) = gof(y)$

Hence, gof is onto.

Q8. Given a non empty set X, consider P(X) which is the set of all subsets of X.

Define the relation R in P(X) as follows: For subsets A, B in P(X), ARB if and only if A ⊂ B. Is R an equivalence relation on P(X)? Justify you answer:

A.8. Since every set is a subset of itself, ARA for all A ∈ P(X).

∴R is reflexive.

Let ARB ⇒ A ⊂ B.

This cannot be implied to B ⊂ A.

For instance, if A = {1, 2} and B = {1, 2, 3}, then it cannot be implied that B is related to A.

∴ R is not symmetric.

Further, if ARB and BRC, then A ⊂ B and B ⊂ C.

⇒ A ⊂ C

⇒ ARC

∴ R is transitive.

Hence, R is not an equivalence relation since it is not symmetric.

Q9. Given a non-empty set X, consider the binary operation *: P(X) × P(X) → P(X) given by A * B = A ∩ B &mn For E; A, B in P(X) is the power set of X. Show that X is the identity element for this operation and X is the only invertible element in P(X) with respect to the operation*.

A.9. Let S be a non-empty set and P(S) be its power set. Let any two subsets A and B of S.

It is given that: $P(X)xP(X) \to P(X)$ is defined as $A.B = A \cap B \forall A, B \in P(X)$

We know that $A \cap X = A = X \cap A \forall A \in P(X)$

$\Rightarrow A.X = A = X.A \forall A \in P(X)$

Thus, X is the identity element for the given binary operation*.

Now, an element is $A \in P(X)$ invertible if there exists $B \in P(X)$ such that

$A * B = X = B * A$ (As X is the identity element)

i.e.

$A \cap B = X = B \cap A$

This case is possible only when $A = X = B$.

Thus, X is the only invertible element in P(X) with respect to the given operation*.

Hence, the given result is proved.

Q10. Find the number of all onto functions from the set {1, 2, 3, … , n) to itself.

A.10. Onto functions from the set {1, 2, 3, … ,n} to itself is simply a permutation on n symbols 1, 2, …, n.

Thus, the total number of onto maps from {1, 2, … , n} to itself is the same as the total number of permutations on n symbols 1, 2, …, n, which is n!.

Q11. Let S = {a, b, c} and T = {1, 2, 3}. Find F^{-1} of the following functions F from S to T, if it exists.

(i) F = {(a, 3), (b, 2), (c, 1)} (ii) F = {(a, 2), (b, 1), (c, 1)}

A.11. S = {a, b, c}, T = {1, 2, 3}

(i) F: S → T is defined as:

F = {(a, 3), (b, 2), (c, 1)}

⇒ F (a) = 3, F (b) = 2, F(c) = 1

Therefore, F^{-1}: T → S is given by

F^{-1} = {(3, a), (2, b), (1, c)}.

(ii) F: S → T is defined as:

F = {(a, 2), (b, 1), (c, 1)}

Since F (b) = F (c) = 1, F is not one-one.

Hence, F is not invertible i.e., F^{-1} does not exist.

Q12. Consider the binary operations*: R ×R → and o: R × R → R defined as a * b = |a - b| and ao b = a, &mn For E; a, b ∈ R. Show that * is commutative but not associative, o is associative but not commutative. Further, show that &mn For E;a, b, c ∈ R, a*(b o c) = (a * b) o (a * c). [If it is so, we say that the operation * distributes over the operation o]. Does o distribute over *? Justify your answer.

A.12. It is given that*: $R \times R \rightarrow$ and $o: R \times R \rightarrow R$ is defined as

$a * b = |a - b| and, aob = a, \&mnForE; a, b \in R.$

For $a, b \in R$, we have:

$$a*b=|a-b|$$
$$b*a=|b-a|=|-(a-b)|=|a-b|$$
$$\therefore a*b=b*a$$

$\therefore$ The operation* is commutative.

It can be observed that,

$$(1.2).3=(|1-2|).3=1.3=|1-3|=2$$
$$1*(2*3)=1*(|2-3|)=1*1=|1-1|=0$$
$$\therefore (1*2)*3=1*(2*3)(where,1,2,3\in R)$$

$\therefore$ The operation* is not associative.

Now, consider the operation o:

It can be observed that $1o2=1, and, 2o1=2.$

$$\therefore 1o2\neq 2o1(where,1,2\in R)$$

$\therefore$ The operation o is not commutative.

Let, $a,b,c\in R$. Then we have:

$$(aob)oc=aoc=a$$
$$ao(boc)=aob=a$$
$$\Rightarrow (aob)oc=ao(boc)$$

$\therefore$ The operation o is associative.

Now, $a,b,c\in R$. Then we have:

$$a*(boc)=a*b=|a-b|$$
$$(a*b)o(a*c)=(|a-b|)o(|a-c|)=|a-b|$$
$$Hence, a*(boc)=(a*b)o(a*c).$$
$$Now,$$
$$1o(2o3)=1o(|2-3|)=1o1=1$$
$$(1o2)*(1o3)=1*1=|1-1|=0$$
$$\therefore 1o(2o3)\neq(1o2)*(1o3)(where,1,2,3\in R)$$

$\therefore$ The operation o does not distribute over*.

Q13. Given a non-empty set X, let *: P(X) × P(X) → P(X) be defined as A * B = (A − B) ∪ (B − A), &mn For E; A, B ∈ P(X). Show that the empty set Φ is the identity for the operation *

and all the elements A of P(X) are invertible with A−1 = A. (Hint: (A − Φ) ∪ (Φ − A) = A and (A − A) ∪ (A − A) = A * A = Φ).

A.13. : It is given that ∗: P(X) × P(X) → P(X) be defined as

A * B = (A – B) ∪ (B – A), A, B ∈ P(X).

Now, let A ϵ P(X). Then, we get,

A * ф = (A – ф) ∪ (ф –A) = A∪ф = A

ф * A = (ф - A) ∪ (A - ф) = ф∪A = A

A * ф = A = ф * A, A ϵ P(X)

Therefore, ф is the identity element for the given operation *.

Now, an element A ϵ P(X) will be invertible if there exists B ϵ P(X) such that

A * B = ф = B * A. (as ф is an identity element.)

Now, we can see that A * A = (A –A) ∪ (A – A) = ф∪ф = ф AϵP(X).

Therefore, all the element A of P(X) are invertible with A-1 = A.

Q14. Define binary operation * on the set {0, 1, 2, 3, 4, 5} as $a*b=\begin{cases} a+b & if, a+b<6 \\ a+b-6 & f, a+b\geq 6 \end{cases}$

Show that zero is the identity for this operation and each element a ≠ 0 of the set is invertible with 6 - abeing the inverse of a

A.14. Let X={0, 1, 2, 3, 4, 5}.

The operation* on X is defined as:

$$a*b=\begin{cases} a+b & if, a+b<6 \\ a+b-6 & if, a+b\geq 6 \end{cases}$$

An element $e\in X$ is the identity element for the operation*, if $a*e=a=e*a\forall a\in X$

For $a\in X$ we observed that

$$a*0=a+0=a\left[a\in X\Rightarrow a+0<6\right]$$
$$0*a=0+a=a\left[a\in X\Rightarrow 0+a<6\right]$$
$$\therefore a*0=0*a\forall a\in X$$

Thus, 0 is the identity element for the given operation*.

An element $a\in X$ is invertible if there exists $b\in X$ such that $a*0=0*a.$

$$ie\begin{cases} a+b=0=b+a & if, a+b<6 \\ a+6-6=0=b+a-6 & if, a+b\geq 6 \end{cases}$$

i.e.,

$a = -b, or, b = 6 - a$

But, X={0, 1, 2, 3, 4, 5} and $a, b \in X$. Then, $a \neq -b$.

$\therefore b = 6 - a$ is the inverse of $a \& mnForE; a \in X$.

Hence, the inverse of an element $a \in X, a \neq 0$ is 6-a i.e., $a^{-1} = 6 - a$.

*	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Q15. Let A = {–1, 0, 1, 2}, B = {–4, –2, 0, 2} and f; g: A→B be the functions defined by *f*(*x*) = $x^2 - x, x \in$ A and g(x) = 2x - 1/2 - 1. x∈ A. Are f and g equal? Justify your answer.

(Hint: One may note that two functions *f*: *A* → *B* and g: *A* → B such that *f*(*a*) = g(*a*) & mn For E; *a* ∈*A*, are called equal functions).

A.15.

It is given that A = {–1, 0, 1, 2}, B = {–4, –2, 0, 2}

Also, it is given that $f, g : A \to B$ are defined by $f(x) = x^2 - x, x \in A$ and

$g(x) = 2x - \frac{1}{2} - 1, x \in A.$

It is observed that:

$f(-1)=(1^2)-(-1)=1+1=2$

$g(-1)=2(-1)-\frac{1}{2}-1=2\left(\frac{3}{2}\right)-1=3-1=2$

$\Rightarrow f(-1)=g(-1)$

$f(0)=(0)^{\wedge}2-0=0$

$g(0)=2(0)-\frac{1}{2}-1=2\left(\frac{1}{2}\right)-1=1-1=0$

$\Rightarrow f(0)=g(0)$

$f(1)=(1)^{\wedge}2-1=1-1=0$

$g(1)=2a-\frac{1}{2}-1=2\left(\frac{1}{2}\right)-1=1-1=0$

$\Rightarrow f(1)=g(1)$

$f(2)=(2)^{\wedge}2-2=4-2=2$

$g(2)=2(2)-\frac{1}{2}-1=2\left(\frac{3}{2}\right)-1=3-1=2$

$\Rightarrow f(2)=g(2)$

$\therefore f(a)=g(a)\forall a\in A$

Hence, the functions f and g are equal.

Q16. Let A = {1, 2, 3}. Then number of relations containing (1, 2) and (1, 3) which are reflexive and symmetric but not transitive is:

(A) 1

(B) 2

(C) 3

(D) 4

A.16. It is clear that 1 is reflexive and symmetric but not transitive.

Therefore, option (A) is correct.

Q17. Let A = {1, 2, 3}. Then number of equivalence relations containing (1, 2) is:

(A) 1

(B) 2

(C) 3

(D) 4

A.17. 2, Therefore, option (B) is correct.

Q18. Let R → R be the Signum Function defined as $f(x)=\begin{cases}1 & x>0\\ 0 & x=0\\ -1 & x<0\end{cases}$ **and R → R be the Greatest Function given by g(x) = [x] where [x] is greatest integer less than or equal to x Then, does fog and gof coincide in (0, 1)?**

A.18. It is given that,

$f:R\to R$ is defined as $f(x)=\begin{cases}1 & x>0\\ 0 & x=0\\ -1 & x<0\end{cases}$

Also, $g:R\to R$ is defined as $g(x)=[x]$, where [x] is the greatest integer less than or equal to x.

Now, let $x\in(0,1)$

Then, we have:

$[x]=1$ if $x=1$ and $[x]=0$ if $0<x<1$

$\therefore\ fog(x)=f(g(x))=f([x])=\begin{cases}f(1) & if, x=1\\ f(0) & if, x\in(0,1)\end{cases}=\{(1,"if, x=1"),(0,: if, x\in(0,1)"):\}$

$gof(x)=g(f(x))$

$=g(1)[x>0]$

$=[1]=1$

Thus, when $x\in(0,1)$, we have $fog(x)=0 and, gof(x)=1$.

Hence, fog and gof do not coincide in (0, 1).

Therefore, option (B) is correct.

Q19. Number of binary operations on the set {a, b} are:

(A) 10

(b) 16

(C) 20

(D) 8

A.19.

A binary operation * on {a, b} is a function from {a, b} × {a, b} → {a, b}

i.e., * is a function from {(a, a), (a, b), (b, a), (b, b)} → {a, b}.

Hence, the total number of binary operations on the set {a, b} is 2^4 i.e., 16.

The correct answer is B.