Miscellaneous Exercise

1. Find the angle between the lines whose direction ratios are a, b, c and $b-c$, $c-a$, $a-b$.

Answer

The angle Q between the lines with direction cosines, a, b, c and $b-c, c-a,$ $a-b,$ is given by,

$$\cos Q = \left| \frac{a(b-c)+b(c-a)+c(a-b)}{\sqrt{a^2+b^2+c^2}+\sqrt{(b-c)^2+(c-a)^2+(a-b)^2}} \right|$$

$$\Rightarrow \cos Q = 0$$

$$\Rightarrow Q = \cos^{-1} 0$$

$$\Rightarrow Q = 90^0$$

Thus, the angle between the lines is 90°

2. Find the equation of a line parallel to x-axis and passing through the origin.

The line parallel to x-axis and passing through the origin is x-axis itself.

Let A be a point on x-axis. Therefore, the coordinates of A are given by $(a, 0, 0)$, where $a \in R$. Direction ratios of OA are $(a-0) = a, 0, 0$

The equation of OA is given by,

$$\frac{x-0}{a} = \frac{y-0}{0} = \frac{z-0}{0}$$

$$\Rightarrow \frac{x}{1} = \frac{y}{0} = \frac{z}{0} = a$$

Thus, the equation of line parallel to x-axis and passing through origin is

$$\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$$

3. If the lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$ are perpendicular, find the value of k.

Answer

The direction of ratios of the lines, $\frac{x-3}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ & $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$, are $-3,\ 2k,\ 2\ and\ 3k,\ 1,\ -5$ respectively.

It is known that two lines with direction ratios, $a_1, b_1, c_1\ and\ a_2, b_2,\ c_2$, are perpendicular, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

$\therefore -3(3k) + 2k \times 1 + 2(-5) = 0$

$\Rightarrow -9k + 2k - 10 = 0$

$\Rightarrow 7k = -10$

$\Rightarrow k = \frac{-10}{7}$

Therefore, for k= -10/7, the given lines are perpendicular to each other.

4. Find the shortest distance between lines $\vec{r} = 6\hat{i} + 2\hat{j} + 2\hat{k} + \lambda(\hat{i} - 2\hat{j} + 2\hat{k})$ and $\vec{r} = -4\hat{i} - \hat{k} + \mu(3\hat{i} - 2\hat{j} - 2\hat{k})$.

Answer

A.9. The given lines are

$\vec{r} = 6\hat{i} + 2j + 2k + \lambda(\hat{i} - 2j - 2k)$..........(1) $\vec{r} = 6\hat{i} + 2j + 2k + \lambda(\hat{i} - 2j - 2k)$..........(1)

$\vec{r} = -4\hat{i} - k + \mu(3\hat{i} - 2j - 2k)$...........(2) $\vec{r} = -4\hat{i} - k + \mu(3\hat{i} - 2j - 2k)$...........(2)

It is known that the shortest distance between two lines, $\vec{r} = a_1 + \lambda b_1\ \&\ \vec{r} = a_2 + \lambda b_2$ is given by

$$d = \left|\frac{(b_1 \times b_2).(a_1 - a_2)}{|b_1 \times b_2|}\right|$$

Comparing $\vec{r} = a_1 + \lambda b_1\ \&\ \vec{r} = a_2 + \lambda b_2$ to equations (1) and (2), we obtain

$\vec{a_1} = 6\hat{i} + 2j + 2k$

$\vec{b_1} = \hat{i} - 2j - 2k$

$\vec{a_2} = -4\hat{i} - k$

$\vec{b_2} = 3\hat{i} - 2j - 2k$

$\Rightarrow \vec{a_2} - \vec{a_1} = (-4\hat{i} - k) - (6\hat{i} + 2j + 2k) = 10\hat{i} - 2j - 3k$

$$\Rightarrow \vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & j & k \\ 1 & -2 & 2 \\ 3 & -2 & -2 \end{vmatrix} = (4+4)\hat{i} - (-2-6)j + (-2+6)k = 8\hat{i} + 8j + 4k$$

$\therefore |\vec{b_1} \times \vec{b_2}| = \sqrt{(8)^2 + (8)^2 + (4)^2} = 12$

$(\vec{b_1} \times \vec{b_2}).(\vec{a_2} - \vec{a_1}) = (8\hat{i} + 8j + 4k).(10\hat{i} - 2j - 3k) = -80 - 16 - 12 = -108$

Substituting all the values in equation (1), we obtain

$$d = \left|\frac{-108}{12}\right| = 9$$

Therefore, the shortest distance between the two given lines is 9 units.

5. Find the vector equation of the line passing through the point (1, 2, – 4) and perpendicular to the two lines:

$$\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7} \text{ and } \frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}.$$

Answer

Let the required line be parallel to the vector $\vec{b}$ given by, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$

The position vector of the point (1, 2, − 4) is $\vec{a}=\hat{i}+2\hat{j}-4\hat{k}$

The equation of the line passing through (1, 2, −4) and parallel to vector $\vec{b}$ is

$$\vec{r}=\vec{a}+\lambda\vec{b}$$
$$\Rightarrow \vec{r}=\left(\hat{i}+2\hat{j}-4\hat{k}\right)+\lambda\left(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}\right)\text{.......(1)}$$

The equations of the lines are

$$\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7}\text{........(2)}$$
$$\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}\text{........(3)}$$

Line (1) and line (2) are perpendicular to each other.

$$\therefore 3b_1-16b_2+7b_3=0\text{........(4)}$$

Also, line (1) and line (3) are perpendicular to each other.

$$\therefore 3b_1+8b_2-5b_3=0\text{........(5)}$$

From equations (4) and (5), we obtain

$$\frac{b_1}{(-16)(-5)-8\times 7}=\frac{b_2}{7\times 3-3(-5)}=\frac{b_3}{3\times 8-3(-16)}$$
$$\Rightarrow \frac{b_1}{24}=\frac{b_2}{36}=\frac{b_3}{72}$$
$$\Rightarrow \frac{b_1}{2}=\frac{b_2}{3}=\frac{b_3}{6}$$

Direction ratios of $\vec{b}$ are 2, 3, and 6.

$$\therefore \vec{b}=2\hat{i}+3\hat{j}+6\hat{k}$$

Substituting $\vec{b}=2\hat{i}+3\hat{j}+6\hat{k}$ in equation (1), we obtain

$$\vec{r}=\left(\hat{i}+2\hat{j}-4\hat{k}\right)+\lambda\left(2\hat{i}+3\hat{j}+6\hat{k}\right)$$

This is the equation of the required line.